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MATHEMATICAL THEORY OF FINANCE

BY

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PREFACE

This book has been prepared primarily to meet the needs of students in schools and colleges of Commerce and Business Administration. It will also meet the needs of anyone who desires a knowledge of the mathematical treatment of financial problems arising in ordinary business procedure.

The scope and method of the book have been designed for a three-hour course for one semester, such as is prescribed in the College of Commerce in the University of California. It is assumed that the student has had a substantial course in Algebra equivalent to two years' study in the high school and a thorough knowledge of logarithmic computation.

The author has aimed throughout to emphasize fundamental principles and to illustrate them with numerous simple examples. Experience in teaching the subject has clearly demonstrated that, in the short time that can usually be allotted to the course, the student can hardly hope to obtain much more than an understanding of basic principles. The more technical phases of theory and application should, the author feels, be left to later study.

The author wishes to acknowledge his indebtedness to his colleagues, Dr. A. R. Williams and Dr. C. D. Shane, for valuable suggestions and criticisms, and in particular to Professor B. A. Bernstein, who has greatly aided both in the preparation of the manuscript and in the correction of the proof.

T. M. Putnam.

University of California,
April, 1923.

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MATHEMATICAL THEORY OF FINANCE

CHAPTER I

INTEREST

1. Definition of interest.—Interest may be defined as money paid for the use of borrowed capital, or, as that which is earned by the productive investment of capital. In a given transaction, the capital involved is referred to as the *principal.*

2. The rate of interest.—The *rate* of interest is the sum earned on a unit of principal in a unit of time, the latter, unless otherwise specified, being taken as one year. It is customary, however, to express the rate as the earning *per centum,* that is, the earning on 100 units of principal. Thus, if each dollar of capital earns 6 cents a year, the rate is 6 *per centum,* meaning that \$100 would earn \$6 per year. Throughout the remainder of this book, the abbreviated form, "per cent," will be used instead of the Latin *per centum.*

3. Simple interest.—Interest usually becomes due at stated intervals and, being of the same nature as capital, may be reinvested as additional principal. If, however, one is concerned only with the amount earned by the original capital in a given time, and not with the productive reinvestment of earnings, the investment is said to bear *simple interest.*

With a given principal and fixed rate, simple interest is proportional to the time. Thus, if P denote the principal, i the rate, and n the time in years, then the interest I is given by the formula

$$I = Pni. \qquad \qquad (1)$$

The *amount*, S, is the sum of the principal and interest, hence

$$S=P+I=P(1+ni). \qquad \ldots\ldots \qquad (2)$$

4. Ordinary and exact simple interest.—In calculating simple interest for fractional parts of a year, it is frequently the practice to base the computation on 360 days to a year. When this is done, interest is called *ordinary simple interest.* If the calculations are based on 365 days in a year, it is called *exact simple interest.*

The exact number of days between two given dates may be calculated directly, or found from a table such as Table I, for the computation of either exact or ordinary simple interest.

Example 1.—Find the ordinary and the exact interest on \$1500 for 80 days at 6 per cent.

The interest for one year is \$90. Hence the ordinary simple interest is

$$\$90\times\tfrac{80}{360}=\$20.$$

The exact simple interest is

$$\$90\times\tfrac{80}{365}=\$19.73.$$

Example 2.—Find the time that has elapsed between March 20, 1921, and October 17, 1921.

From Table I it is found that October 17 is the 290th day of the year and March 20 is the 79th day of the year; hence there are 211 days between these dates.

Unless otherwise stated, it will be understood that ordinary simple interest is required.

PROBLEMS

1. What is the monthly simple interest on a note for \$2500 bearing 5 per cent? *Ans.* \$10.42.

2. Find the time a note for \$1850, bearing 6 per cent simple interest, would have to run in order to amount to \$2000. *Ans.* 1 yr., 4 mo., 6 days.

3. Find the ordinary simple interest on \$1250 at 7 per cent, from April 10 to August 25 of the same year. *Ans.* \$33.30.

4. Find the exact simple interest in Problem 3. *Ans.* \$32.84.

5. Find the ratio of exact interest to ordinary interest, showing that it is constant for any number of days. *Ans.* $\frac{72}{73}$

6. What is the rate of interest, if $1800 earns $45 in 4 months?

7. What principal will amount to $1263 in 10 months, 15 days, at 6 per cent?

8. Find the exact simple interest on a note for $1500 bearing 6 per cent interest, dated February 4, 1920 (a leap year) and falling due November 20, 1920. Find also the ordinary simple interest.

5. Compound interest.—It was seen that simple interest is calculated on the original principal only, and is merely proportional to the time. If interest, when due, is added to the principal, and interest for the next period is calculated on the principal thus increased, this process being continued with each succeeding accumulation of interest, then the interest is said to be *compound.*

It should be observed that, in transactions involving simple interest paid at regular intervals, the creditor, collecting his interest and investing it at the same rate as in the original loan, will accumulate new capital just as rapidly as if he had loaned at compound interest originally.

Because interest is itself of the nature of capital, it becomes necessary, in all questions involving equivalence of value, to regard all sums as bearing compound interest. This is particularly important when an indebtedness extends over a considerable length of time.

6. Formulas of compound interest.—Let P be the principal, and i the rate of interest. The amount to which the principal will accumulate will be denoted by S. The interest for one year will be Pi, and the amount at the end of that time will be $P+Pi$. This becomes the principal for the second year. The interest on it is $(P+Pi)i$, and the amount at the end of the second year will be the principal plus the interest earned, or

$$P+Pi+(P+Pi)i=P(1+i)^2.$$

Thus, each unit of principal at the beginning of any year will accumulate to $1+i$ units at the end of the year, so that the amount may be obtained by multiplying the principal at the beginning of the year by $1+i$. Since $P(1+i)^2$ is the principal at the beginning of the third year, the amount at the end of the third year will be $P(1+i)^3$. In general, the amount, S, at the end of n years, is given by the formula

$$S=P(1+i)^n. \qquad (3)$$

In this process, interest is said to be *compounded*, or *converted*, annually. It may, however, be compounded semiannually, or quarterly, or, in general, m times a year. The time between two successive conversions of interest into principal is spoken of as the *conversion period*.

The same principles as used above apply when the period is a fraction of a year, provided merely that i is replaced by the interest on one unit of principal for the period in question, and n is replaced by the total number of conversion periods. Thus, if the rate of interest is 2 per cent per half year, the amount of \$100 for 10 years is $\$100(1.02)^{20}$. It is customary to speak of 2 per cent per half year as "4 per cent converted semiannually," so that in general if interest is at rate j converted m times a year, Formula (3) is replaced by

$$S=P\left(1+\frac{j}{m}\right)^{mn}. \qquad (4)$$

In deriving Formulas (3) and (4), the time is supposed to contain an integral number of conversion periods. When this is not so, it is customary in practice to compute the amount at the end of the last conversion period, and then to compute simple interest for the fraction of a period remaining. For theoretical purposes, however, it is convenient to regard Formulas (3) and (4) as true for all values of n, whether integers or not.

EXAMPLE.—Find the amount at compound interest on \$1000, at 4 per cent converted semiannually, for 2 years and 8 months. Formula (4) becomes

$$S=\$1000(1.02)^{16/3}$$
$$=\$1111.39$$

The number of conversion periods is 5. Had the amount been computed at the end of the fifth period, it would have been

$$\$1000(1.02)^5 = \$1104.08.$$

If simple interest be computed on this sum for the remaining 2 months, it is seen to be \$7.36. The final amount thus computed is then \$1111.44, which is slightly in excess of that given by the first method.

Unless otherwise stated, it will be understood that the amount, S, is to be computed by Formulas (3) or (4), whether or not the time contains an integral number of periods.

Table II gives the values of $(1+i)^n$ for the usual rates of interest that arise in applying either Formula (3) or Formula (4). Thus, for example, if the rate of interest is 5 per cent compounded quarterly, there will be $4n$ periods, while $\frac{j}{m}$, the rate of interest for the quarterly period, will be $1\frac{1}{4}$ per cent. If n were 5 years, the amount of 1 would be found in the column headed $1\frac{1}{4}$ per cent, and opposite the number 20 in the left-hand column. The problem is identical with finding the amount of 1 for 20 years at compound interest at $1\frac{1}{4}$ per cent per year.

For rates of interest not listed in the table, and for values of n that are not integers, the computations may be made by use of logarithms.

7. Geometrical comparison of simple and compound interest. —If the amount, S, be plotted on an ordinary coordinate system, time being laid off on the horizontal scale and the corresponding values of S vertically, the simple interest formula

$$S = P(1+ni)$$

is a straight line (Fig. 1). The compound interest amount, $S=P(1+i)^n$, gives a curve which is concave upward, intersecting the simple interest line at $n=0$ and $n=1$. For times less than a year, the amount by compound interest is less than that by simple interest, but after one year the compound amount is larger, on account of the addition of interest to principal.

When interest is compounded m times a year, the curve corresponding to

$$S = P\left(1+\frac{j}{m}\right)^{mn}$$

bears a corresponding relation to the simple interest line, intersecting it at $n=0$ and at $n=\frac{1}{m}$.

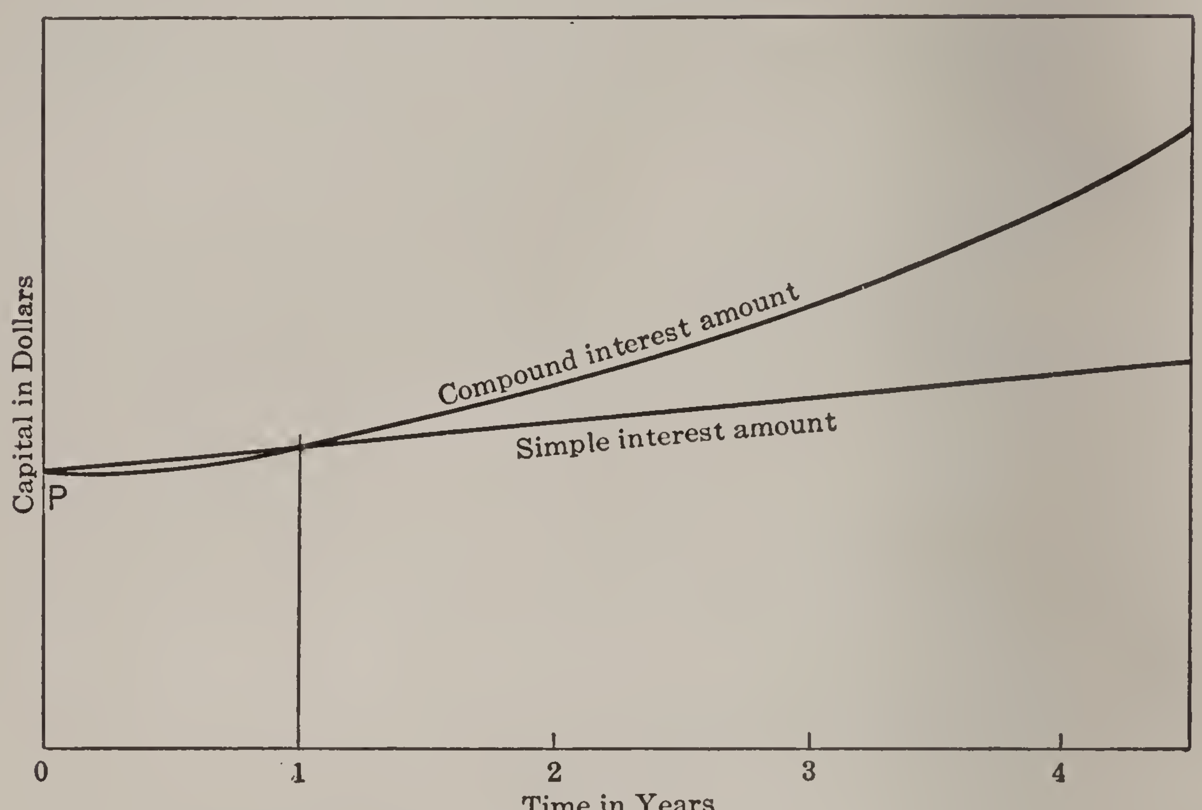

FIG. 1.

PROBLEMS

1. Using logarithms, find the amount of \$100 for 5 years, compounded annually, at 4 per cent. How many places should be given in the tables used, if the result is to be correct to the nearest cent? Compare your answer with the result obtained from Table II.

2. How long would it take \$1 to double at 4 per cent, compounded annually? At 8 per cent?

3. Find the amount of \$1000 for 5 years, 8 months, at 5 per cent, compounded semiannually. Compute this amount also by using the tables for the 11 whole periods involved and computing simple interest for the fractional period at the end.

4. A father, at the birth of his son, sets aside a sum that will amount to \$5000 in 21 years. If it earns 4 per cent, compounded semiannually, what is the sum? *Ans.* \$2176.52.

5. If the population of a town in 1920 was 4526, and its annual rate of increase during the previous decade was 8 per cent, what will its population be in 1925, if this rate continues?

6. Show how to use the tables to obtain the value of $(1+i)^n$ for values of n outside the range of the tables. Find the amount of \$1, for 35 years, compounded quarterly, at 6 per cent per annum.

7. Using (3), calculate the interest on \$1000, at 5 per cent, for 6 months. Compare it with the simple interest for the same period.

8. Draw the graphs showing the amounts, by simple interest and by compound interest, for \$100 bearing 6 per cent, compounded annually.

8. Nominal and effective rates of interest.—When interest is compounded more frequently than once a year, the rate of interest quoted is called the *nominal* rate. Thus, if interest is at 6 per cent per annum, compounded quarterly, the nominal rate is 6 per cent, but the rate for the quarterly conversion period is $1\frac{1}{2}$ per cent. It is customary to indicate a nominal rate of interest by the letter j. If interest is converted m times a year, the symbol $j_{(m)}$ may be used to designate both the nominal rate and the frequency of compounding. Thus, if $j_{(2)}=0.04$, interest is being compounded at 2 per cent per half year.

The *effective* rate of interest is the amount earned by each unit of capital during one year. It will be denoted by the letter i. Thus, if $j_{(m)}$ is the nominal rate of interest,

$$1+i=\left(1+\frac{j_{(m)}}{m}\right)^m. \qquad (5)$$

From (5) one obtains

$$i=\left(1+\frac{j_{(m)}}{m}\right)^m-1 \qquad (6)$$

and

$$j_{(m)}=m\{(1+i)^{\frac{1}{m}}-1\}. \qquad (7)$$

It should be noted that $j_{(1)}=i$, but that, for m greater than 1, i will always be larger than $j_{(m)}$. Thus, if $j_{(4)}=0.06$,

$$i=(1.015)^4-1=0.061364.$$

On the other hand, if $i=0.06$, Formula (7) gives

$$j_{(4)}=4\{(1.06)^{1/4}-1\}=0.058695.$$

The reason for these relations becomes apparent when one considers that, when interest is compounded more frequently than once a year, the interest, thus added to the principal itself earns interest, which increases the annual earnings and thus makes the effective rate larger than the nominal rate.

Table VIII gives the values of $j_{(2)}$, $j_{(4)}$ and $j_{(12)}$ for rates of interest used in the tables of this book. When a nominal rate of interest is given, Table II may be used, together with Formula (6), in order to determine the corresponding effective rate.

PROBLEMS

1. Find the effective rate of interest when money is compounded semi-annually at 5 per cent.

2. What nominal rate, interest compounded quarterly, will produce an effective rate of 8 per cent?

3. If a business grows from $12,000 to $16,000 in 5 years, what is the effective rate of interest involved?

4. Given, $i=0.06$; compute $j_{(2)}$, $j_{(4)}$, $j_{(12)}$ and $j_{(365)}$ to four decimal places.

5. Which is the better investment, one in which interest is at 6 per cent, compounded quarterly, or one in which simple interest is earned at $6\frac{1}{8}$ per cent per annum? Compare the earnings in the two cases for one year on $1000 principal.

9. Continuous compound interest. Force of interest.—In the relation (7)

$$j_{(m)}=m\{(1+i)^{\frac{1}{m}}-1\},$$

it was seen that, for a given effective rate, i, $j_{(m)}$ will diminish as m, the number of conversions per year is increased. The length of the interest period will then diminish, and, as m increases without bound, one obtains a state of affairs in which interest may be thought of as being compounded continuously. The limit that $j_{(m)}$ approaches as m approaches infinity is called the *force of interest.* It is denoted by the letter δ. Hence,

$$\delta = \lim_{m=\infty} m[(1+i)^{\frac{1}{m}} - 1].$$

Expanding $(1+i)^{\frac{1}{m}}$ by the binomial theorem, and simplifying,

$$\delta = \lim_{m=\infty}\left[i + \frac{\frac{1}{m}-1}{2}i^2 + \frac{\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)}{2.3}i^3 + \ldots\right].$$

$$= i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \ldots.$$

The latter series, however, is the expansion of $\log_e (1+i)$, where $e = 2.71828\ldots$, the base of the natural system of logarithms. Hence,

$$\delta = \log_e (1+i), \qquad (8)$$

and

$$1+i = e^{\delta}. \qquad (9)$$

Example.—Find the force of interest when $i=0.06$ (see Problem 4, § 8).

$$\delta = \log_e 1.06$$

$$= \frac{\log_{10} 1.06}{\log_{10} e}$$

$$= 0.05827.$$

By Formula (9), the amount, S, is given by

$$S = P(1+i)^n = Pe^{n\delta}. \qquad (10)$$

PROBLEMS

1. The valuation of property in a given community may be thought of as increasing continuously. If in 1920 it was \$10,000,000, and the rate of continuous growth is constant, $\delta = 0.05$, what will the valuation be in 1930?

2. What is the force of interest corresponding to a nominal rate $j_{(4)} = 0.08$?

3. Find the amount of \$1000 for 20 years at 6 per cent, compounded annually. What would the amount be if interest were compounded continuously at 6 per cent ($\delta = 0.06$)?

10. Present value.—In § (6) it was seen that P units of capital, put at interest now at rate i, will amount to S units in n years, where

$$S = P(1+i)^n.$$

This means also that a promise to pay S dollars in n years could be discharged equitably by the payment of P dollars now. For this reason, P is called the *present value* of S. The term *present* is used in a technical sense, for one may compute the "present value" of a sum S, n years before it is due, which date may or may not be the "present" time for the computer. The term, however, may be justified by imagining the present to be the date at which the value is to be computed.

It is customary to represent by the letter v the present value of 1 due in one year. Hence,

$$1 = v(1+i),$$

or

$$v = \frac{1}{1+i}, \quad \ldots \quad (11)$$

so that

$$P = S\frac{1}{(1+i)^n} = Sv \quad \ldots \quad (12)$$

The quantity v is always less than 1; hence, the larger n is, the smaller will v^n be, and therefore, the more remote the date at which S is due, the smaller P will be. This, of course, is otherwise obvious, from the nature of the relation between P and S.

The values of powers of v are given in Table III for usual rates of interest.

If interest is compounded m times a year and the corresponding nominal rate $j_{(m)}$ is given, then the fundamental relation (5) gives

$$v = \frac{1}{1+i} = \frac{1}{\left(1+\frac{j_{(m)}}{m}\right)^m}.$$

Hence, in terms of $j_{(m)}$,

$$P=S\frac{1}{\left(1+\frac{j_{(m)}}{m}\right)^{mn}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When $j_{(m)}$ is given, Table III may still be used to find P, if the rate of interest $\frac{j_{(m)}}{m}$ is one of those listed; otherwise, P would have to be found by use of logarithms.

As an example showing the use of the table, suppose it is desired to find the present value of \$1000 due in 5 years, where interest is compounded quarterly at the nominal rate of 7 per cent, *i.e*, $j_{(4)}=0.07$. Formula (13) gives

$$P=1000\frac{1}{(1.0175)^{20}}.$$

This is the same as finding the present value of \$1000 due in 20 years, interest being allowed at the rate of $1\frac{3}{4}$ per cent. From the table, the value of $(1.0175)^{-20}$ is found to be 0.7068246; hence,

$$P=\$1000\times 0.7068246,$$

$$=\$706.82.$$

PROBLEMS

1. Find the present value of \$1000 due in 10 years, if money is worth 6 per cent effective.

2. Find the present value of \$1000 due in 10 years, if money is worth 6 per cent, converted semiannually. (Note that $P=1000v^{20}$ at 3 per cent.)

3. What sum should be deposited in a savings bank paying 4 per cent, compounded semiannually, in order that in 10 years it may amount to \$6000?

4. Which is the better offer for a piece of property: (*a*) \$2000 cash and \$1000 at the end of each year for 3 years; or (*b*) \$1250 cash and \$1250 at the end of each year for 3 years? Make a comparison on a 5 per cent basis, finding the equivalent present values.

Ans. (*a*) \$4723.25; (*b*) \$4654.06.

5. Find the present value of $10,000 due in 20 years, interest at the rate of 8 per cent effective. Find the present value with interest at 4 per cent effective.

6. In Problem 5, what would the present value be if $j_{(4)}=0.08$?

11. Discount.—The difference between a sum of money due at a future date and its present value is called discount. The rate of discount is the discount for one year on one unit of principal. If this be denoted by d, then, by the definition,

$$d=1-v. \qquad (14)$$

Replacing v by its value, $\frac{1}{1+i}$,

$$d=\frac{i}{1+i}=iv, \qquad (15)$$

from which also,

$$i=\frac{d}{1-d}. \qquad (16)$$

When the rate of discount is given, it is important to know the corresponding rate of interest. This is given by (16); and, while it is expressed in terms of effective rates, the same formula may be used to give the relation between the rate of interest and the corresponding rate of discount, for fractions of a year. Thus, if a bank discounts a bill due in 90 days, at the nominal rate of 6 per cent per annum, it means that the discount for the period is $1\frac{1}{2}$ per cent. The corresponding rate of interest is $\frac{j_{(4)}}{4}$.

Formula (16) gives, therefore,

$$\frac{j_{(4)}}{4}=\frac{0.015}{0.985}=0.01523.$$

The corresponding effective rate of interest is given by the fundamental relation

$$1+i=\left(1+\frac{j_{(4)}}{4}\right)^4,$$

from which $i=0.0623$. Hence, a person who uses his funds to discount 90-day commercial paper at the nominal rate of 6 per cent is earning nearly $6\frac{1}{4}$ per cent on his money.

PROBLEMS

1. A bill for \$278.50, due in 90 days, was sold for \$270. What was the nominal rate of discount? *Ans.* 12.2 per cent.

2. A salary check for \$300, due on July 1, was discounted June 20 at 7 per cent. How much was deducted?

3. Paper due in 60 days is discounted at the nominal rate of 8 per cent. What is the corresponding nominal rate of interest? *Ans.* 8.1 per cent.

4. Find the effective rate of interest corresponding to a discount rate of 2 per cent per quarter.

5. Derive Formulas (14) and (15) by considering an amount v, invested now and earning d in one year, as interest.

12. Principle of equivalence.—In this chapter it has been made clear that *time* is an important factor in determining the amount of money necessary to discharge a debt. The sum that will clear an obligation to-day will not be sufficient a year from now, or at any other subsequent date. The difference is due to interest, and interest has been seen to be an increasing function of the time. Thus, if money is worth 6 per cent, and no interest is paid in the meantime, a debt of \$1000 to-day can be cleared five years from now only by the payment of

$$\$1000(1.06)^5=\$1338.23.$$

Under the assumed interest rate, it may be said that \$1000 to-day is *equivalent* to \$1338.23 five years hence. The same debt could be cleared two years hence by the payment of \$1123.60; it could have been cleared one year ago by the payment of \$943.40. All of these sums are equivalent.

As a further illustration, suppose that one debt is to be discharged by the payment of \$1000 one year hence, and another by the payment of \$1500 in two years. What amount could be paid at the end of eighteen months to discharge both obligations equitably, if money is worth 5 per cent?

This problem can be solved by letting X be the required payment and equating it to the sum of the *equivalent* values of the two debts. Thus,

$$X = \$1000(1.05)^{\frac{1}{2}} + \$1500(1.05)^{-\frac{1}{2}}$$
$$= \$1024.70 + \$1463.85$$
$$= \$2488.55.$$

The same result would have been obtained had the equivalent sums been computed at any other date and the corresponding equation formed. Thus, if the present values had been found, the result would have been

$$Xv^{\frac{3}{2}} = 1000v + 1500v^2,$$

or,

$$X = 1000v^{-\frac{1}{2}} + 1500v^{\frac{1}{2}},$$

which gives the same value as found above.

In the succeeding chapters, the principle of equivalence of values, under the compound interest law, will be fundamental. This principle is, indeed, directly or indirectly involved in nearly every financial problem.

MISCELLANEOUS PROBLEMS

1. Find the exact simple interest on \$200 for 73 days at 7 per cent. Compare it with the interest as found by using (3), putting $n = 0.2$.
Ans. \$2.80; \$2.73.

2. How long will it take \$1 to double itself at 6 per cent, compounded annually? How long if compounded quarterly?

3. Find the amount of \$1 at 4 per cent interest, compounded semi-annually, for 100 years.

4. Construct a graph showing the amount, when \$1 bears interest at 5 per cent effective. Calculate the amounts for every half year for 5 years.

5. What is the effective rate of interest when 1 per cent per month is charged ($j_{(12)} = 0.12$)?

6. Find the force of interest corresponding to an effective rate of 4 per cent.

7. Find the nominal rate of interest realized, if a bill for $500, due in 90 days, is discounted for $490.

8. If funds are utilized to discount 60-day paper at 6 per cent nominal, what effective rate of interest is realized?

9. If money is worth 7 per cent, what sum, paid one year hence, will equitably discharge two obligations, one due in 9 months for $250, the other due in 18 months for $400, each without interest.

10. An obligation is to be discharged 3 years hence, by the payment of $3000. Find the amount of each of two equal payments, one to be made 1 year hence and the other 2 years hence, that will be the equivalent, if money is worth 6 per cent. *Ans.* $1373.88.

11. If $300 is due in 30 days, $250 in 90 days, and $600 in 180 days, all sums without interest, at what time could the total, $1150, be paid in one sum to discharge these debts equitably, money being worth 6 per cent

CHAPTER II

ANNUITIES

13. Definition.—A series of equal payments, made at equal intervals of time, is called an *annuity*. The word implies yearly payments, but the term is used to describe any series of payments, made at equal intervals of time, which may be of any length. Unless otherwise stated, the payments are understood to be made at the *end* of each interval and to continue for a specified number of periods.

14. Notation.—Given any transaction involving equal periodic payments, two important questions immediately arise: to find, under given interest conditions, (1) the *present value* of all the payments, and (2) the *amount* of all the payments accumulated to the end of the last period. These computations are based on an annuity whose total annual payment is 1.

The following symbols are used: $a_{\overline{n|}}$ denotes the present value of an annuity of 1 per annum for n years, the total payment of 1 being made in one installment at the end of each year.

$a_{\overline{n|}}^{(p)}$ denotes the present value of an annuity of 1 per year for n years, the annual payment, however, being made in p equal installments of $\frac{1}{p}$, at the end of each pth part of a year.

$s_{\overline{n|}}$ denotes the amount of an annuity of 1 per annum for n years, and $s_{\overline{n|}}^{(p)}$ the amount when the annual payment is made in p equal installments, at the end of each pth part of a year.

15. Present value of an annuity.—By the definition,

$$a_{\overline{n|}} = v + v^2 + c^3 + \dots + v^n, \qquad \dots \dots \quad (1)$$

being merely the present value of each payment of 1 due at the end of each of the n years. The right member of (1) is a geometrical progression whose common ratio is v. Its sum is, therefore,

$$a_{\overline{n}|}=\frac{v-v^{n+1}}{1-v}=\frac{1-v^n}{\frac{1}{v}-1},$$

But $\frac{1}{v}=1+i$; hence,

$$a_{\overline{n}|}=\frac{1-v^n}{i}. \qquad (2)$$

If the annual payment is R instead of 1, and A denotes its present value, the formula becomes

$$A=R\cdot a_{\overline{n}|}=R\frac{1-v^n}{i}. \qquad (3)$$

Table IV gives the values of $a_{\overline{n}|}$ for ordinary rates of interest.

Formula (2) could also be obtained by direct reasoning, in the following manner. Suppose \$1 to be loaned for n years, at rate i. The lender is entitled to interest, i, each year and to the return of the original dollar at the end of n years. The interest constitutes an annuity of i, whose present value is $ia_{\overline{n}|}$, while the present value of 1 due in n years is v^n. Hence, the original investment of 1 must provide for $ia_{\overline{n}|}$, to take care of the annual interest, and for v^n to be set aside to accumulate to 1 in n years, so as to return the original capital. Hence,

$$1=ia_{\overline{n}|}+v^n,$$

or,

$$a_{\overline{n}|}=\frac{1-v^n}{i}.$$

For values of i, or n, not in the table, $a_{\overline{n}|}$ must be computed by finding v^n by means of logarithms, and then performing the indicated arithmetical operations. Formula (3) contains four quantities, A, R, i, and n. If three are given, the fourth may be determined. Except when i is the unknown, this offers little difficulty.

EXAMPLE 1.—Find the cost of an annuity of \$100 per year for 12 years, allowing interest at $5\frac{1}{2}$ per cent. This rate of interest is not found in the table; hence $a_{\overline{12|}}$ must be computed directly.

$$a_{\overline{12|}} = \frac{1 - \dfrac{1}{(1.055)^{12}}}{0.055}.$$

By using logarithms, $(1.055)^{-12} = 0.52601$. Substituting and simplifying, we find $a_{\overline{12|}} = 8.6180$. Hence the cost of an annuity of \$100 per year is \$861.80.

EXAMPLE 2.—If \$10,000 is paid for an annuity yielding \$800 per year, how many years will it run, if interest is allowed at 5 per cent?
Formula (3) gives

$$10{,}000 = 800a_{\overline{n|}}.$$

Hence,

$$a_{\overline{n|}} = 12.5 \text{ (at 5 per cent).}$$

Therefore,

$$\frac{1-v^n}{.05} = 12.5$$

$$v^n = 0.375 \text{ where } v = (1.05)^{-1}.$$

Therefore,

$$n = \frac{-\log 0.375}{\log 1.05} = 20.10 \text{ years.}$$

As defined, $a_{\overline{n|}}$ requires that n be an integer. The result here, however, may be interpreted as indicating that 20 payments of \$800 may be made, but the payment at the end of the twenty-first year will be less than \$800, to close the transaction equitably. The cost of an annuity of \$800, to run 20 years at 5 per cent, is

$$\$800a_{\overline{20|}} = \$9969.76.$$

The difference between \$10,000 and this sum is \$30.24, which, accumulated to the end of the twenty-first year, amounts to

$$\$30.24(1.05)^{21} = \$84.25,$$

which is therefore the sum to be paid at the end of the twenty-first year.

The time could also be obtained directly from the annuity tables, by noting that, at 5 per cent, $a_{\overline{20|}} = 12.462$, and $a_{\overline{21|}} = 12.821$; hence, the value of n that satisfies the equation, $a_{\overline{n|}} = 12.5$, lies between 20 and 21. Indeed, by interpolation, n is found to be 12.106.

PROBLEMS

1. What is the present value of an annuity of \$100, payable at the end of each year for 10 years, if money is worth 6 per cent? Verify, by direct computation, the value as found from the tables.

Ans. \$736.01.

2. Find the costs of annuities of \$100, to run 15 years, payable in single annual installments, interest being allowed at the following rates, respectively, (*a*) 4 per cent; (*b*) 6 per cent; (*c*) 8 per cent.

Ans. (*a*) \$1111.84; (*b*) \$971.22 (*c*) \$855.95.

3. A man purchases a house, paying \$4000 down and \$600 at the end of each year for 5 years. What would be the equivalent price if he paid all in cash at the time of purchase, money being worth 7 per cent?

Ans. \$6460.11.

4. A piece of property is purchased for \$25,000, the purchaser paying \$5000 down and agreeing to pay the balance, with interest at 7 per cent, in annual installments of \$2500. How long will it take to clear the transaction, and how large will the last payment be?

Ans. 13 years; \$345.27.

5. How much should be paid for a mine that can be made to yield \$15,000 net per year for 10 years, after which it will be worthless. The income is supposed to be available at the end of each year, and the investment is to yield 8 per cent to the investor. *Ans.* \$100,651.22.

6. What is the present value of an annuity of \$1000, to run 8 years, interest being allowed at $7\frac{1}{2}$ per cent?

16. Formula for $a_{\overline{n|}}^{(p)}$.—In this case the annual payment of 1 is made in p equal installments of $\frac{1}{p}$ each. Suppose that interest be converted in agreement with these payments and at the nominal rate $j_{(p)}$; then formula (3) may be applied where a periodic payment of $\frac{1}{p}$ is made for np periods, interest being allowed at the rate of $\frac{j_{(p)}}{p}$ per period. Hence,

$$a_{\overline{n|}}^{(p)} = \frac{1}{p}\, a_{\overline{np|}} \left(\text{at rate } \frac{j_{(p)}}{p}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

This may be expressed in terms of the effective rate of interest, i, by means of the fundamental relation

$$1+i=\left(1+\frac{j_{(p)}}{p}\right)^p.$$

Substituting in (4) the value of $a_{\overline{np}|}$,

$$a_{\overline{n}|}^{(p)}=\frac{1}{p}\,\frac{1-\left(1+\dfrac{j_{(p)}}{p}\right)^{-np}}{\dfrac{j_{(p)}}{p}}$$

$$=\frac{1-(1+i)^{-n}}{j_{(p)}}.$$

Hence, finally

$$a_{\overline{n}|}^{(p)}=\frac{1-v^n}{j_{(p)}}=\frac{i}{j_{(p)}}a_{\overline{n}|}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Formula (5) should be used when i is given. The factor $\dfrac{i}{j_{(p)}}$ can be obtained from Table IX for the usual values of i, and for $p=2$, 4 and 12, corresponding to semi-annual, quarterly and monthly payments.

PROBLEMS

1. Find the cost of an annuity of \$1000 per year, to run 20 years, (*a*) if payable in one installment and $i=0.04$; (*b*) if payable in two installments and the corresponding $j_{(2)}=0.04$; (*c*) if payable in four installments and the corresponding $j_{(4)}=0.04$.

Ans. (*a*) \$13,590.33; (*b*) \$13,677.19; (*c*) \$13,722.05.

2. Find the cost of an annuity of \$400, payable in quarterly installments of \$100, to run 8 years, interest at 8 per cent nominal ($j_{(4)}=0.08$).

Ans. \$2346.83.

3. What would the cost be in Problem 2, if interest were at 8 per cent effective? *Ans.* \$2366.49.

4. Find the cost of an annuity of \$100 per month, to run 15 years, interest at 4 per cent effective.

5. What is the present value of an annuity that pays \$1000 every quarter, to run 12 years, interest at 5 per cent effective?

6. Find the present value of an annuity paying \$50 per month for 20 years, interest at $3\frac{1}{2}$ per cent effective.

7. What is the cost of an annuity paying \$600 each half year for 10 years, interest at 4 per cent nominal ($j_{(2)}=0.04$)? What would the cost be if interest were at 4 per cent effective ($i=0.04$)?

17. Formulas for $s_{\overline{n}|}$ and $s_{\overline{n}|}^{(p)}$.—From the definitions (§ 14) of $s_{\overline{n}|}$ and $s_{\overline{n}|}^{(p)}$, it is seen that they are merely the values of the series of payments accumulated to the end of the n years. But $a_{\overline{n}|}$ and $a_{\overline{n}|}^{(p)}$ represent the values of these same sums at the beginning of the n years; hence,

$$s_{\overline{n}|}=a_{\overline{n}|}(1+i)^n, \qquad (6)$$

$$s_{\overline{n}|}^{(p)}=a_{\overline{n}|}^{(p)}(1+i)^n \qquad (7)$$

Substituting the values of $a_{\overline{n}|}$ and of $a_{\overline{n}|}^{(p)}$ in (6) and (7) respectively,

$$s_{\overline{n}|}=\frac{(1+i)^n-1}{i}; \qquad (8)$$

and

$$s_{\overline{n}|}^{(p)}=\frac{(1+i)^n-1}{j_{(p)}}. \qquad (9)$$

For purposes of computation, (9) may be transformed as follows:

$$s_{\overline{n}|}^{(p)}=\frac{(1+i)^n-1}{i}\cdot\frac{i}{j_{(p)}}$$

$$=s_{\overline{n}|}\cdot\frac{i}{j_{(p)}}. \qquad (10)$$

Table V gives the values of $s_{\overline{n}|}$ for the usual values of i and n, and Table IX gives the values of $\frac{i}{j_{(p)}}$ for $p=2, 4, 12$.

When the annual payment is R instead of 1, the amount S is given by the formulas

$$S=Rs_{\overline{n}|}, \text{ or } S=Rs_{\overline{n}|}^{(p)}. \qquad (11)$$

When the payments are made in p installments per year, and a nominal rate of interest, $j_{(p)}$, is given, interest being compounded in agreement with payments, then the problem becomes one of finding the amount of an annuity of $\frac{1}{p}$ per period, for np periods, at rate $\frac{j_{(p)}}{p}$. Hence,

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} \cdot s_{\overline{np}|}\left(\text{at rate } \frac{j_{(p)}}{p}\right). \quad . \quad . \quad . \quad . \quad (12)$$

Formula (8) can also be derived by the following reasoning An investment of 1, with its accumulations of interest, amounts to $(1+i)^n$ in n years. The annual interest, however, constitutes an annuity, which amounts to $is_{\overline{n}|}$ at the end of n years. This, with the original dollar, then equals $(1+i)^n$, *i.e.*,

$$1+is_{\overline{n}|} = (1+i)^n;$$

hence,

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

PROBLEMS

1. If \$1000 is invested at the end of each year for 20 years, at 4 per cent, find the amount at the end of the period.

Find the amount if \$500 is invested at the end of every 6 months for 20 years, interest being allowed at 4 per cent nominal ($j_{(2)} = 0.04$).

2. Compare the amounts of annuities of \$100 at 5 per cent, running, respectively, 5, 10, 15 and 20 years.

3. Compare the amounts of annuities running 10 years, each for \$100, at 4 per cent, 6 per cent and 8 per cent, respectively.

4. Use the $s_{\overline{n}|}$ table to find how long it will take a man to accumulate \$10,000, by putting \$300 in a savings bank every 6 months, interest at 4 per cent nominal, converted semiannually.

Ans. At the end of 13 years amount is \$10,101.27.

5. Find the amount of an annuity paying \$150 per quarter, accumulated 12 years at 6 per cent effective.

6. If \$100 is placed in a savings bank at the end of each month, for 5 years, and interest is allowed at 4 per cent effective, how much will be on deposit at the end of the period?

7. Find the amount of an annuity of \$250 per quarter at the end of 5 years, at 5 per cent effective.

8. Find the amount of an annuity of \$25 per month for 15 years, interest at 6 per cent nominal ($j_{(12)}=0.06$). What would the amount be if the rate of interest were 6 per cent effective ($i=0.06$)?

18. Deferred annuities.—A deferred annuity is one whose payments do not begin until after a certain period of years has elapsed.

The symbols for the present value of an annuity of 1, deferred m years, are $m|a_{\overline{n}|}$ and $m|a_{\overline{n}|}^{(p)}$, according as the payments are made once, or p times a year.

The amounts of deferred annuities, after they have run n years, are clearly the same as the amounts of ordinary annuities for n years.

The present value of an annuity deferred m years is the same as the present value of an annuity to run $m+n$ years, diminished by the cost of an annuity for m years. Hence,

$$m|a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|},$$

$$m|a_{\overline{n}|}^{(p)} = a_{\overline{m+n}|}^{(p)} - a_{\overline{m}|}^{(p)}. \qquad (13)$$

These formulas are adapted to computation, the values of the quantities in the right members being readily obtained by use of the tables.

It should be noted that another expression for $m|a_{\overline{n}|}$ is given by

$$m|a_{\overline{n}|} = v^m a_{\overline{n}|},$$

because $a_{\overline{n}|}$ represents the value of the annuity when the payments are to begin, and $v^m \cdot a_{\overline{n}|}$ is its present value.

19. Annuities due.—When the payments of an annuity are made at the beginning of each period, instead of at the end, it is called an *annuity due*. The present value of an annuity due of 1 is indicated by the symbol $\mathbf{a}_{\overline{n}|}$, and its amount by $\mathbf{s}_{\overline{n}|}$. If the payments are made p times a year the respective symbols are $\mathbf{a}_{\overline{n}|}^{(p)}$ and $\mathbf{s}_{\overline{n}|}^{(p)}$.

Aside from the first payment of 1, the annuity due, to run

n years, is equivalent to an ordinary annuity to run $n-1$ years. Hence,

$$\mathbf{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \qquad (14)$$

Similarly,

$$\mathbf{a}_{\overline{n}|}^{(p)} = \frac{1}{p} + a_{\overline{n-\frac{1}{p}}|}^{(p)}. \qquad (15)$$

The amount of an annuity due, $\mathbf{s}_{\overline{n}|}$, may be found by considering each payment carried forward to the end of its period, when it will amount to $1+i$. The annuity due is then equivalent to an ordinary annuity of $1+i$ per annum, running n years. Hence,

$$\mathbf{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}. \qquad (16)$$

Similarly,

$$\mathbf{s}_{\overline{n}|}^{(p)} = (1+i)^{\frac{1}{p}} s_{\overline{n}|}^{(p)}. \qquad (17)$$

One can also think of $\mathbf{s}_{\overline{n}|}$ as the amount of an ordinary annuity running $n+1$ years, with the last payment omitted, so that

$$\mathbf{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1. \qquad (18)$$

Similarly,

$$\mathbf{s}_{\overline{n}|}^{(p)} = s_{\overline{n+\frac{1}{p}}|}^{(p)} - \frac{1}{p}. \qquad (19)$$

PROBLEMS

1. What sum of money should be set aside now in order to provide \$1200 a year for 4 years, the first payment to be made 18 years hence, interest at 4 per cent effective? *Ans.* \$2236.19.

2. Find the cost of an annuity of \$100 per month, deferred 10 years and to run 8 years, interest at 5 per cent effective. *Ans.* \$4869.50.

3. How much money should be set aside on Jan. 1, 1923, in order to accumulate to a sufficient amount to provide an annuity of \$1200, payable in quarterly installments of \$300, the first to be paid on April 1, 1930, the last on Jan. 1, 1935, all sums to bear interest at a rate $j_{(4)} = 0.04$?

4. One thousand dollars is put in a savings bank on Jan. 1, 1920, and a like sum every 6 months thereafter until July 1, 1930, inclusive. If interest is allowed at 4 per cent, compounded semiannually, how much will be on deposit after the last payment?

5. One hundred dollars is placed in a savings bank at the beginning of each month for 6 years. Simple interest on all balances is allowed at 4 per cent, but this is compounded semiannually. Show that this is equivalent to an annuity of \$607 at 2 per cent, running 12 years. Find the accumulated amount.

6. Prove that $\mathbf{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.

7. Find the cost of an annuity due, the annual payment of which is \$400, to run 12 years, interest at 5 per cent effective.

8. Find the cost of an annuity due, paying \$100 per quarter, to run 12 years, interest at 5 per cent nominal ($j_{(4)} = 0.05$).

20. The annuity that 1 will purchase.—The annual income, R, from an annuity whose present value is A, is found by solving (3) for R, giving

$$R = \frac{A}{a_{\overline{n}|}}. \qquad (20)$$

If, in a particular case, we let $A = 1$, we have

$$R = \frac{1}{a_{\overline{n}|}}, \qquad (21)$$

as the income from an annuity that 1 will purchase.

If the annuity is payable p times a year, then the income from an investment of 1 is

$$\frac{1}{a_{\overline{n}|}^{(p)}} = \frac{j_{(p)}}{i} \cdot \frac{1}{a_{\overline{n}|}}. \qquad (22)$$

The values of $\frac{1}{a_{\overline{n}|}}$ can be found from Table VI for ordinary rates of interest. If the annuity is payable in p installments, and the effective rate, i, is given, formula (22) will be used. The value of $j_{(p)}$ can be found from Table VIII for $p = 2, 4$ and 12.

If $j_{(p)}$ is given instead of i, then from (4) we have

$$\frac{1}{a_{\overline{n}|}^{(p)}} = \frac{p}{a_{\overline{np}|}} \cdot \left(\text{at rate } \frac{j_{(p)}}{p}\right). \qquad (23)$$

Thus, the income from an annuity costing 1 is obtained; the annuity costing A will produce A times as much.

21. The annuity that will amount to 1.—The annual payment, R, necessary to give an annuity that will accumulate to 1 in n years, is found by putting $S=1$ in (11), giving

$$R=\frac{1}{s_{\overline{n}|}},\quad \text{or}\quad R=\frac{1}{s_{\overline{n}|}^{(p)}}, \quad \ldots \ldots \quad (24)$$

according as the annuity is payable in one installment, or in p installments, respectively. Since, from (12) and (10),

$$s_{\overline{n}|}^{(p)}=\frac{1}{p}\,s_{\overline{np}|}\quad\left(\text{at rate }\frac{j_{(p)}}{p}\right),$$

and

$$s_{\overline{n}|}^{(p)}=\frac{i}{j_{(p)}}\,s_{\overline{n}|}\quad(\text{at rate }i).$$

then,

$$\frac{1}{s_{\overline{n}|}^{(p)}}=\frac{p}{s_{\overline{np}|}}\quad\left(\text{at rate }\frac{j_{(p)}}{p}\right), \quad \ldots \ldots \quad (25)$$

or,

$$\frac{1}{s_{\overline{n}|}^{(p)}}=\frac{j_{(p)}}{i}\,\frac{1}{s_{\overline{n}|}}\ (\text{at rate }i). \quad \ldots \ldots \quad (26)$$

The values of $\frac{1}{s_{\overline{n}|}}$ may be obtained from the tables for $\frac{1}{a_{\overline{n}|}}$ by means of the simple relation

$$\frac{1}{a_{\overline{n}|}}=\frac{1}{s_{\overline{n}|}}+i. \quad \ldots \ldots \ldots \quad (27)$$

This may be established by direct reasoning. An investment of 1 produces an annuity whose annual yield is $\frac{1}{a_{\overline{n}|}}$. On the other hand, from an investment of 1, there should be the annual interest i and in addition a sufficient sum $\frac{1}{s_{\overline{n}|}}$ which, set aside annually, will amount to 1 at the end of n years, thus returning the original capital.

Hence, $\frac{1}{s_{\overline{n}|}}$ may be obtained from the corresponding value of

$\frac{1}{a_{\overline{n}|}}$ in Table VI, by merely subtracting from it the rate of interest.

By an analogous argument,

$$\frac{1}{a^{(p)}_{\overline{n}|}}=\frac{1}{s^{(p)}_{\overline{n}|}}+j_{(p)}. \qquad (28)$$

EXAMPLE 1.—Find the annual yield of a 10-year annuity payable in quarterly installments, interest at 4 per cent nominal, ($j_{(4)}=0.04$), purchased for $10,000.

From (23) the annual yield is

$$R=\$10{,}000\,\frac{1}{a^{(4)}_{\overline{10}|}}=\$40{,}000\,\frac{1}{a_{\overline{40}|}} \text{ (at 1 per cent).}$$

From Table VI,

$$\frac{1}{a_{\overline{40}|}}=0.0304556,$$

thence

$$R=\$1218.22.$$

EXAMPLE 2.—How much should be paid annually in order that the accumulation in 10 years may be $2000, interest at 5 per cent effective?

From (27),

$$\frac{1}{s_{\overline{10}|}}=\frac{1}{a_{\overline{10}|}}-0.05$$

$$=0.0795046,$$

whence,

$$R=\$2000\,\frac{1}{s_{\overline{10}|}}=\$159.01.$$

PROBLEMS

1. A house is purchased for $15,000, and it is arranged that $5000 cash be paid, and the balance in 10 equal annual installments, including interest at 6 per cent. Find the annual payment.

2. A debt of $3000 is to be paid off by 36 equal monthly installments, including interest at 5 per cent effective. What is the monthly payment?

3. What sum, invested every 6 months at 4 per cent, compounded semi-annually, will amount to $5000 in 10 years?

4. If $j_{(4)}=0.06$, what is the quarterly payment necessary to accumulate to \$3000 in 5 years?

5. Prove algebraically that $\dfrac{1}{a_{\overline{n}|}}=\dfrac{1}{s_{\overline{n}|}}+i.$

22. Perpetuities and capitalization.—When an annuity is continued for an unlimited period, it is called a *perpetuity.*

The present value of a perpetuity of 1 is clearly $\dfrac{1}{i}$, because this is the amount of capital necessary to produce 1 per annum as interest.

This also may be deduced from the relation

$$a_{\overline{n}|}=\frac{1-v^n}{i},$$

by letting n increase. For, since v is less than 1, the limit of v^n, as n increases without bound, is zero.

Another form of perpetuity occurs when regular payments have to be made for an indefinite period, but at intervals of several years. For example, a certain part of a plant may have to be renewed every k years. A question would arise as to how much capital should be set aside, in order to provide, through its interest earnings, funds sufficient to pay for these renewal charges. If x denotes this capital, then xi would be the annual interest, and if S is the amount to be raised every k years, then,

$$xi\, s_{\overline{k}|}=S,$$

or,

$$x=\frac{S}{i}\cdot\frac{1}{s_{\overline{k}|}}. \qquad (29)$$

The quantity given by (29), when added to the first cost, S, is called the *capitalized cost* of the article.

It is clear that it may also be found by summing the series

$$S+Sv^k+Sv^{2k}+Sv^{3k}+\ldots,$$

an infinite geometrical progression whose sum is

$$\frac{S}{1-v^k}=\frac{S}{i}\cdot\frac{1}{a_{\overline{k}|}}. \qquad \ldots \ldots \ldots \quad (30)$$

But, from (27),

$$\frac{1}{a_{\overline{k}|}}=\frac{1}{s_{\overline{k}|}}+i;$$

hence, the right member of (30) may be replaced by

$$S+\frac{S}{i}\cdot\frac{1}{s_{\overline{k}|}},$$

which is the first cost plus the present value of an indefinite number of renewals.

For purposes of computation, Formula (30) should be used.

Formula (29) may also be obtained by reasoning as follows: The capital, x, that is to be set aside to provide for the renewals, will, in k years, amount to $x(1+i)^k$. At that time, a sum S is to be withdrawn, after which the original capital, x, should remain, to be allowed to accumulate for another k years. Hence,

$$x(1+i)^k-S=x,$$

or

$$x=\frac{S}{(1+i)^k-1}=\frac{S}{i}\cdot\frac{1}{s_{\overline{k}|}}.$$

Example.—Compare the capitalized costs of two machines, on a 6 per cent basis, one costing \$2500 and lasting 5 years, the other costing \$4000 but good for 9 years. If both are capable of doing the same work, which is the better investment?

$$\frac{\$2500}{0.06}\;\frac{1}{a_{\overline{5}|}}=\$9891.52.$$

The capitalized cost of the second is

$$\frac{\$4000}{0.06}\cdot\frac{1}{a_{\overline{9}|}}=\$9801.48.$$

The latter, therefore, offers a slight advantage over the former.

MISCELLANEOUS PROBLEMS

1. A man owes \$1000, and is to pay it in monthly installments of \$20, with interest at 6 per cent nominal ($j_{(12)}=0.06$). How long will it take? How much is due just after the last full payment of \$20 is made?

2. Ten thousand dollars is invested in an annuity of \$60 per month, interest at 6 per cent effective. How long does it run?

3. If \$1000 is placed in a fund at the end of each year, interest at 7 per cent effective, what will it amount to in 7 years, 11 months?

4. If \$100 is annually placed in a fund drawing interest at 5 per cent effective, how long will it have to run before it will be sufficient to buy an annuity of \$1000 per year for 10 years?

5. Find the cost of an annuity of \$750, deferred 12 years, interest at 6 per cent effective.

6. Find the cost of an annuity of \$200 per year, payable in quarterly installments, the rate of interest being 5 per cent effective, the payments to run 15 years.

7. A man pays \$6000 for a mine; he sets aside \$1500 at the beginning of each year, for 3 years, for development work, and, at the beginning of the fourth year, \$3000 for a mill. How much should the mine produce annually, beginning with the fifth year, in order to net him 8 per cent effective on the whole investment, if the value of the mine is exhausted at the end of the tenth year?

8. Two hundred dollars per month is put in a savings bank on the first of each month, beginning January, 1920. Simple interest at 4 per cent is allowed on all balances on deposit, and the accumulated interest is added to the principal on July 1 and January 1 of each year. How much will be on deposit July 2, 1925?

9. If it costs \$1500 a year to maintain a certain number of dirt tennis courts, how much could be spent to cover them with asphalt, to be equivalent, if upkeep is thus reduced to \$300 per year, interest at 6 per cent?

10. Prove algebraically that

$$\frac{1}{a^{(p)}_{\overline{n}|}}=\frac{1}{s^{(p)}_{\overline{n}|}}+j_{(p)}.$$

11. What single payment, made in advance, is equivalent to \$100 paid at the end of each month for 12 months, if money is worth 6 per cent effective?

12. How much must be paid now for an annuity of \$250 per quarter, the first payment to be made $5\frac{1}{4}$ years hence, interest at the rate $j_{(4)}=0.05$, the payments to terminate after 20 have been made?

13. Compute the value of $a_{\overline{n}|}$ by regarding it as a sum of money drawing interest at rate i, compounded annually, but from which 1 is withdrawn at the end of each year for n years, at which time the fund is exhausted.

14. How long would it take to pay off a debt of \$800 by making monthly payments of \$20, allowing interest at 6 per cent nominal, i.e., $j_{12}=0.06$? How large is the last payment?

15. If a nominal rate $j_{(m)}$ be given instead of $j_{(p)}$, then, since from (7),

$$j_{(p)}=p\{(1+i)^{\frac{1}{p}}-1\},$$

and since

$$1+i=\left(1+\frac{j_{(m)}}{m}\right)^m,$$

$$j_{(p)}=p\left\{\left(1+\frac{j_{(m)}}{m}\right)^{\frac{m}{p}}-1\right\}.$$

Hence, from (5),

$$a_{\overline{n}|}^{(p)}=\frac{1-\left(1+\frac{j_{(m)}}{m}\right)^{-mn}}{p\left\{\left(1+\frac{j_{(m)}}{m}\right)^{\frac{m}{p}}-1\right\}}.$$

16. Use the result of Problem 15 to obtain the cost of an annuity of \$800, payable in quarterly installments for 10 years, ($j_2=0.06$).

17. Obtain the formula for $a_{\overline{n}|}^{(p)}$ by finding the present value of each of the np payments of $\frac{1}{p}$, and summing the resulting geometrical progression.

18. Prove, by direct reasoning, that

$$m|a_{\overline{n}|}=s_{\overline{n}|}\cdot v^{m+n}=a_{\overline{n}|}\cdot v^m.$$

19. From the tables, find approximately the rate of interest, if an annuity of \$100 amounts to \$3492.58 in 20 years.

20. Find the formula for $s_{\overline{n}|}$, by finding the amount of each payment at the end of n years at rate i, and adding them. The sum is a geometrical progression.

21. When a rate of interest $j_{(m)}$, is given, show that

$$s_{\overline{n}|}^{(p)} = \frac{\left(1+\frac{j_{(m)}}{m}\right)^{mn}-1}{p\left[\left(1+\frac{j_{(m)}}{m}\right)^{\frac{m}{p}}-1\right]}.$$

(See problem 15.)

22. Use the result of Problem 21 to find the amount of an annuity of \$250, paid in quarterly installments of \$62.50 each, running 8 years, interest being allowed at 4 per cent, converted semiannually.

CHAPTER III

AMORTIZATION—SINKING FUNDS

23. Amortization.—The extinction of debts by uniform periodic payments occurs frequently in financial transactions, and gives rise to an important application of annuities.

If K represents a debt, then the annual payment, R, necessary to extinguish K in n years and to pay interest charges, is the annual payment of an annuity that K will buy. Hence,

$$R = K \cdot \frac{1}{a_{\overline{n|}}}. \qquad (1)$$

The general process by which the principal of a debt is repaid by periodic payments is called *amortization*. The term, however, will, in this chapter, be limited to the method just described, whereby the debtor makes *equal* periodic payments, which include both interest and a partial return of principal. The interest charge decreases as the principal is reduced; consequently, as time goes on, an increasing amount of the fixed periodic payment is applied to the reduction of principal. This fact is illustrated in the next article.

24. Amortization schedules.—Consider a debt of \$1000 bearing 6 per cent interest. Suppose that it is desired to repay this in 10 equal annual installments, including interest.

From (1) the annual payment will be

$$\begin{aligned} R &= \$1000 \frac{1}{a_{\overline{10|}}} \quad \text{(at 6 per cent)}, \\ &= \$135.87. \end{aligned}$$

Interest for the first year will be \$60; hence, \$75.87 of the first payment would be applied to the reduction of principal, leaving

\$924.13 due at the beginning of the second year. The interest on this amount for a year is \$55.45; hence the principal is reduced by \$80.42 by the second payment. The continuation of the process may be readily traced in the following table, which may be called the *amortization schedule.*

Year.	Principal at Beginning of Year.	Interest at 6 Per Cent.	Principal Repaid.
1	\$1000.00	\$60.00	\$ 75.87
2	924.13	55.45	80.42
3	843.71	50.62	85.25
4	758.46	45.51	90.36
5	668.10	40.09	95.78
6	572.32	34.34	101.53
7	470.79	28.25	107.62
8	363.17	21.79	114.08
9	249.09	14.95	120.92
10	128.17	7.69	128.18
		\$358.69	\$1000.01

The total amount paid during the 10 years is \$1358.70, which checks with the sum of the last two columns.

25. Amount of unpaid principal.—It is important to know, at any time, the amount of unpaid principal. Such information is needed in keeping accounts and calculating liabilities, and for the purpose of closing the transaction at an earlier date. This can be learned, of course, from a schedule such as that given in § 24. It can, however, be obtained by simply noting that, if k payments have been made, the remaining $n-k$ constitute an annuity whose present value is the outstanding principal. Denoting this by A_k, then,

$$A_k = Ra_{\overline{n-k|}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where R has the value given by (1).

PROBLEMS

1. Find the annual payment that will be necessary to amortize, in 5 years, a debt of $1000, bearing interest at 7 per cent. *Ans.* $243.89.

2. A man owes 2000 on an automobile, and wishes to pay it off, with interest at 6 per cent nominal, in 15 equal monthly installments ($j_{(12)} = 0.06$). How much should he pay monthly? How much will he still owe at the end of 1 year? *Ans.* $138.73; $412.04.

3. Construct a schedule showing the amortization of a debt of $25,000 in 10 equal semiannual payments, interest at 8 per cent nominal ($j_{(2)} = 0.08$).

4. A debt of $2000, bearing interest at 6 per cent nominal ($j_{(12)} = 0.06$), is being paid by equal monthly installments, running 5 years. How much will still remain due 2 years hence?

5. A debt of $8000 is to be paid in 5 equal annual installments, including interest at 7 per cent. What is the annual payment. if the first is made immediately instead of at the end of the first year?

Ans. $1823.48.

6. A person owes $8000. He arranges to pay it, principal and interest, in 12 equal semiannual installments, interest at 6 per cent nominal ($j_{(2)} = 0.06$). After 8 payments have been made, a new arrangement is agreed upon, whereby the balance is paid in 6 additional equal payments, instead of 4. Find the amount of each of the latter payments.

7. A debt of $2500, with interest at 6 per cent nominal ($j_{(12)} = 0.06$), is being paid off in 30 equal monthly payments. At the end of 2 years, the debtor wishes to pay the balance in cash. How much should he pay, including the last monthly payment?

26. Sinking funds.—When an obligation becomes due at some future date, it is frequently desirable to anticipate the necessary payment by accumulating a fund by periodic contributions, together with interest earnings. This is called a *sinking fund.*

For example, a corporation issues $1,000,000 in 6 per cent bonds, due in 15 years, interest payable semiannually. They pay the $30,000 interest charge every 6 months, but, in addition, wish to set aside, semiannually, a sum sufficient to accumulate to $1,000,000 in 15 years, at which time they must redeem the bonds. Suppose that they can earn only 4 per cent on their sinking fund, compounded semiannually. From (24), § 21, the necessary semiannual payment is seen to be

$$R = \$1{,}000{,}000\,\frac{1}{s_{\overline{30}|}},\ \text{(at 2 per cent)},$$
$$= \$24{,}649.90.$$

The total semiannual payment necessary to take care of this debt, principal and interest, is therefore, \$54,649.90.

If they were to accumulate their sinking fund at 6 per cent, converted semiannually, the total semiannual charge, including bond interest, would be \$51,019.30. This is the same as the amortization charge, as given by (1).

In general terms, the annual payment, R, into a sinking fund which is being accumulated at rate i, and which must amount to K in n years, is given by

$$R = K\cdot\frac{1}{s_{\overline{n}|}},\ \text{(at rate } i). \qquad (3)$$

If the debt K bears interest at rate i', then the annual interest charge is Ki'. If we denote by R' the combined sinking fund and interest payments, we have

$$R' = K\cdot\frac{1}{s_{\overline{n}|}} + Ki'. \qquad (4)$$

But, from (27), § 21, we have,

$$\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i.$$

Hence,

$$R' = K\cdot\frac{1}{a_{\overline{n}|}} + K(i'-i). \qquad (5)$$

If the sinking fund accumulates interest at the same rate as that paid on the debt K, then $i' = i$ and

$$R' = K\cdot\frac{1}{a_{\overline{n}|}}.$$

The total annual payment for interest and sinking fund charge is, therefore, the same in this case as by the amortization method. This was illustrated in the foregoing example.

27. Amount in the sinking fund at any time.—If S_r represents the amount in the sinking fund at the end of r years, its value can be found by computing the amount of an annuity of R per annum, which has run r years.

If K represents the debt and n the number of years before it is due, we have, from (3)

$$R = K \cdot \frac{1}{s_{\overline{n}|}},$$

hence,

$$S_r = R \cdot s_{\overline{r}|},$$

or,

$$S_r = K \cdot \frac{s_{\overline{r}|}}{s_{\overline{n}|}}. \qquad (6)$$

PROBLEMS

1. A debt of \$6000, bearing 7 per cent interest, is due in 4 years. A sinking fund is to be accumulated at 5 per cent effective. What is the annual payment necessary to take care of both interest and sinking fund? *Ans.* \$1812.07.

2. A city has a bonded indebtedness of \$1,000,000, maturing in 20 years. A sinking fund is created, on which 4 per cent is earned, converted semiannually. How much will be in the fund at the end of 10 years?

3. A city with \$40,000,000 assessed valuation issues \$300,000 worth of bonds, redeemable in 25 years and bearing interest at 5 per cent. A sinking fund is created, yielding 4 per cent effective, into which equal annual payments are made. How much will the tax rate of the city be increased to provide interest on the bonds and to pay the sinking fund charge? *Ans.* 5.55 cents on each \$100.

4. What must be the monthly payment into a sinking fund in order to accumulate to \$5000 in 3 years, interest being allowed at the nominal rate $j_{12} = 0.06$? *Ans.* \$127.11.

5. Quarterly payments are being made into a sinking fund on which 5 per cent interest is earned ($j_{(4)} = 0.05$). How much is the quarterly payment if the sinking fund is to amount to \$20,000 in 8 years?

6. A debt of $8000 is to be paid off at the end of 6 years, from a sinking fund earning interest at 4 per cent nominal, converted semiannually. Find the amount of the semiannual payment into this sinking fund if made at the *beginning* of each half year.

28. Amortization of bonded debt.—If it is desired to repay a debt represented by bonds of a given denomination, it is not possible to make the annual payments of principal and interest exactly equal, because the amount paid for reduction of principal must be a multiple of the face value of the bonds. In such cases the amount R, necessary to repay the debt in equal annual payments, is determined as in § 24. After the interest has been deducted from R in any given year, the number of bonds that can be retired with the balance may be determined, and a schedule constructed. If the bonds are bought in the open market, this schedule will have to be carried forward from year to year, in order to be accurate, particularly if there is considerable fluctuation in the prices at which the bonds are bought.

Example.—Construct a schedule showing the retirement of an indebtedness represented by 50 bonds of the face value of $1000, bearing interest at 6 per cent, payable annually. The annual payments for principal and interest are to be as nearly equal as possible, the whole debt to be repaid in 5 years.

If the annual payments were all equal, each would equal

$$R = \$50{,}000 \frac{1}{a_{\overline{5}|}} = \$11{,}869.82.$$

Interest for the first year would be $3000. Subtracting this from R leaves $8869.82. The number of bonds that should be retired the first year is 9, this being the nearest multiple of $1000 that can be used to approximate $8869.82. The following schedule shows the continuation of the process until the end of the 5 years. The total annual payment, shown in the last column, is sometimes larger than R and sometimes smaller, but differs from it always by less than $500. The number of bonds that can be retired each year increases as the interest charge diminishes.

Year.	Principal.	Interest at 6 Per Cent.	No. of Bonds Retired.	Principal Repaid.	Total Annual Payment.
1	$50,000	$3000	9	$9000	$12,000
2	41,000	2460	9	9000	11,460
3	32,000	1920	10	10,000	11,920
4	22,000	1320	11	11,000	12,320
5	11,000	660	11	11,000	11,660
		$9360	50	$50,000	$59,360

PROBLEMS

1. Construct a schedule showing the retirement of an indebtedness represented by 100 bonds of the denomination of $1000, bearing interest at 5 per cent, payable annually. The annual payments for principal and interest are to be as nearly equal as possible, the whole debt to be repaid in 8 years.

2. Construct a schedule for the retirement in 5 years of a debt represented by 1000 bonds, each of par value $100 and bearing 4 per cent interest, payable semiannually. Suppose that bonds are repurchased in the open market at 102. Arrange the schedule so that the amount paid for interest semiannually, together with the amount paid for the bonds which are to be repurchased semiannually at interest dates, shall be as nearly equal as possible.

29. Depreciation.—In the operation of physical property of every kind, there is a deterioration that cannot be provided for by current repairs. Buildings, machinery and equipment of all sorts diminish in value through use and through the mere action of the elements. Buildings may last fifty years or more, while wooden piles in salt water survive only a short time, and machinery a relatively brief period, depending primarily upon use. This loss in value which cannot be made good by current repairs is called *depreciation.*

It is a fundamental principal of economics that capital invested in business enterprises should not be impaired. From current revenues, then, there should be set aside sufficient amounts to replace worn-out articles, or to keep intact the amount of capital originally invested in them. To do this it is necessary to know the probable life of the article and to have an estimate of its residual, or *scrap*, value. The accumulation of a depreciation fund may then be accomplished by the sinking-fund method. A sum can be set aside annually, which at the end of the life of the article will amount to the difference between the original cost and the scrap value. This difference is called the depreciable value, or *wearing value.* At any intermediate date during the life of the article, its *book value* may be taken to be the original cost less the amount in the sinking fund; therefore, the two items taken together preserve the original capital intact at all times.

If C is the cost, S the scrap value, and n the estimated life, the annual depreciation charge will be given by

$$D=(C-S)\frac{1}{s_{\overline{n|}}}=W\cdot\frac{1}{s_{\overline{n|}}}, \quad \ldots \ldots \quad (7)$$

where W denotes the wearing value.

The amount in the depreciation fund at the end of r years is $Ds_{\overline{r|}}$. Hence, the book value at that time is

$$C-Ds_{\overline{r|}}=C-W\frac{s_{\overline{r|}}}{s_{\overline{n|}}}. \quad \ldots \ldots \quad (8)$$

Example.—Suppose that an article costing \$1200 has a scrap value of \$200 at the end of 10 years. It is proposed to accumulate a sinking fund at 4 per cent to replace the capital lost by depreciation, and to regard the book value of the article, at any time during the interval, as equal to the original value of \$1200 diminished by the amount in the sinking fund. Construct a schedule showing the amount in the sinking fund at the end of each year and the resulting book value of the article.

By (3), § 26, the amount that must be paid annually into the sinking fund is

$$D=\$1000\frac{1}{s_{\overline{10|}}}=\$83.29.$$

The amount in the sinking fund at the end of any year, r, can be found by computing $Ds_{\overline{r|}}$, or can be found by calculating the interest on the amounts in the sinking fund year by year and adding it to the sinking fund together with the annual payment. The following table shows the results:

Year.	Book Value at Beginning of Year.	Total Amount in Sinking Fund at End of Yea .	Interest on Sinking Fund
1	$1200.00	$ 83.29	$ 0.00
2	1116.71	169.91	3.33
3	1030.09	260.00	6.80
4	940.00	353.69	10.40
5	846.31	451.13	14.15
6	748.87	552.47	18.05
7	647.53	657.86	22.10
8	542.14	767.46	26.31
9	432.54	881.45	30.70
10	318.55	1000.00	35.26
11	200.00		

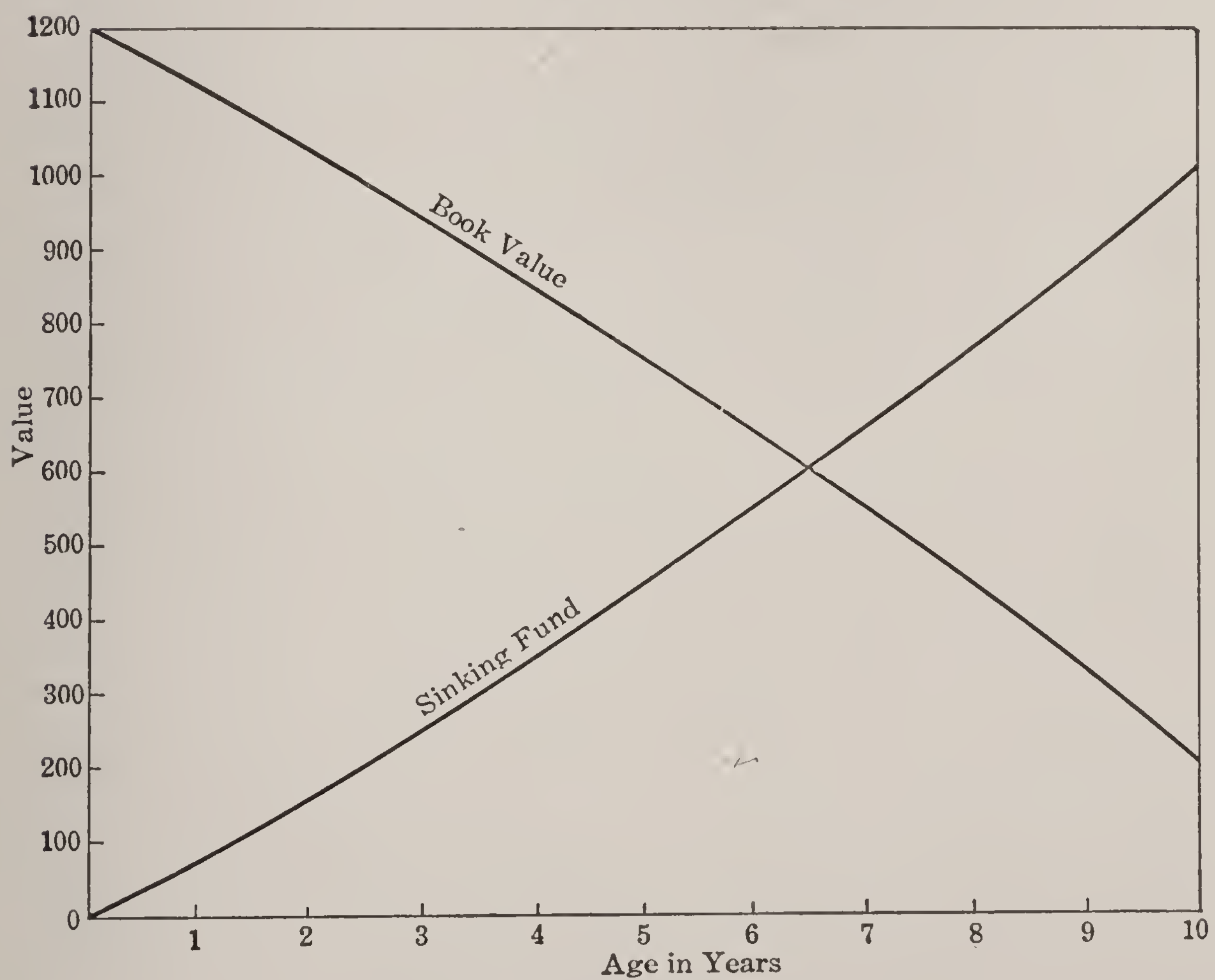

FIG. 2.

In Fig. 2 the growth of the sinking fund and the corresponding decrease in book value, in the preceding example, are illustrated graphically.

PROBLEMS

1. The capital represented by an auto truck costing $1600, with a probable life of 8 years and a scrap value of $200, is to be replaced by money set aside in a 4 per cent sinking fund by equal annual payments. Find the amount of the payment and the book value at the end of 5 years.

2. A plant consists of three parts, with costs, scrap values, and probable lives as given in the following table:

Part.	Cost.	Scrap Value.	Life.
A	$50,000	$5000	25 years
B	20,000	3000	15 years
C	10,000	1000	8 years

Find the total annual payment into a sinking fund, accumulated on a 4 per cent basis that will be necessary in order to provide for depreciation. What will the total book value be at the end of 8 years?

Ans. $2906.29; $53,220.78.

3. A pipe line has a probable life of 15 years. If its wearing value is $100,000, what should the annual depreciation charge be, on a 4 per cent basis? Find the book value at the end of 10 years.

30. Other methods of estimating depreciation.—The sinking-fund method of allowing for depreciation, as just discussed, is only one of many plans used in practice. Its chief advantage is that the annual depreciation charge is constant. The decrease in book value varies, being slightly greater each year than it was in the preceding year, and becoming more rapid toward the end of the life of the article.

Another procedure, known as the *straight-line method,* makes the decrease in book value the same each year. Thus, in the example discussed in § 29, $100 would be "written off" each year. If this amount were set aside without interest it would provide the necessary sum for replacement at the end of 10 years. If it is invested during the interval, the interest earned can properly be turned back into income.

Some corporations follow a plan of estimating depreciation by allowing a fixed percentage of the total valuation each year. The rate is determined, as nearly as possible, so as to provide, with interest accumulations, a fund from which sums can be drawn to provide for replacements as they become necessary. Referring to the example in § 29, let r denote the constant percentage to be deducted each year; then the value at the beginning of the second year would be $\$1200(1-r)$. At the end of the second year it would be $\$1200(1-r)^2$ and so on. If the scrap value at the end of 10 years is \$200, then

$$1200(1-r)^{10}=200.$$

By use of logarithms, r may be found from this equation to be equal to 16.405 per cent.

In the general case, r is determined by the equation

$$C(1-r)^n=S. \qquad \qquad (9)$$

As in the case of the straight-line method, the amount written off each year may be put aside in a sinking fund, and any interest earned returned to the general revenue account.

For other methods of estimating depreciation, the student is referred to books on accounting.

PROBLEMS

1. If an article, which is worth \$2400, when it is new, depreciates in 8 years to a scrap value of \$400, construct a schedule showing its book value by the straight-line method of estimating depreciation.

2. What constant percentage would have to be written off each year if that method were used in the preceding example? Construct a schedule showing the book value for each year.

3. A building costing \$100,000 is estimated to have a life of 50 years, with a residual value of \$10,000. Construct and compare schedules showing the book values as computed by the straight-line and by the percentage methods.

31. Composite life.—If a business concern contains several parts of different probable lives, the *composite life* is defined as the time required for the total depreciation charge to accumulate to the total wearing value. Thus, if W_1, W_2, . . ., W_k are the wearing values of the individual parts, and n_1, n_2, . . . n_k, their respective duration; then, if

$$W = W_1 + W_2 + \dots + W_k,$$

and

$$D = D_1 + D_2 + \dots + D_k,$$

where D_1, D_2, . . . D_k represents the annual depreciation charge for the various parts; then, by the straight-line method

$$n = \frac{W}{D}.$$

By the sinking-fund plan

$$Ds_{\overline{n}|} = W,$$

or

$$s_{\overline{n}|} = \frac{W}{D}. \qquad (10)$$

The value of n may be determined by the use of logarithms, or from the $s_{\overline{n}|}$ tables by locating the nearest value of $s_{\overline{n}|}$ in the appropriate interest column. Thus, in Problem 2, § 29, we have $W = \$71,000$ and $D = \$2906.29$; whence,

$$s_{\overline{n}|} = 24.43 \text{ (at 4 per cent).}$$

From the $s_{\overline{n}|}$ tables, n is found to be between 17 and 18 years. By interpolation, $n = 17.36$ years.

The composite life of a plant is an average of the lives of its individual parts, weighted according to their costs, and with interest taken into account. Considered in another way, it is the life of a hypothetical object whose wearing value is equal to the combined wearing values of all the parts of the plant in question, and which is being provided for by the annual depreciation payment. The knowledge of its value is useful in certain questions arising in finance, such as the determination of the term of a bond issue loaned on the property as security. Such an issue would ordinarily be for a term not exceeding the composite life of the plant, on the principle that money borrowed to purchase equipment, or other adjunct of a business, and on these articles as security, at least in part, should be repaid before the articles are worn out.

PROBLEM

Find the composite life of a plant consisting of the following parts:

Part.	Cost.	Scrap Value.	Life in Years.
A	\$100,000	\$10,000	40
B	25,000	2,000	25
C	15,000	1,000	15
D	12,000	500	7

(*a*) Interest at 6 per cent. (*b*) Interest at 4 per cent.

32. Valuation of mining property.—In operating a mine, a quarry, or any similar property in which there is a limited product, sooner or later to be exhausted, provision must be made to restore, or keep intact, the capital invested in the enterprise. This is done by setting aside part of the annual income as a redemption fund.

If the mining engineer is able to make fairly accurate estimates of the total amount of mineral in the mine, and of the cost and rate at which it can be removed, the value of the mine can be readily computed. Its value is the present value of an annuity yielding the estimated net annual revenue, and to run the number of years given by the engineer's estimate.

Thus, if the annual revenue is R and its life n years, then its value V is given by the formula

$$V = Ra_{\overline{n|}} = R\,\frac{1-v^n}{i}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where i is the investment rate. The annual interest on the investment is Vi; hence,

$$R - Vi = Rv^n$$

is available out of the annual income as a payment into the redemption fund. If this can be accumulated at rate i, it will amount in n years to

$$Rv^n s_{\overline{n}|} = Ra_{\overline{n}|} = V,$$

thus restoring the original capital.

However, on account of certain risks involved in such enterprises, the investor usually demands a relatively higher investment rate of interest than he can expect to obtain on money paid into the redemption fund. If i' denotes the investment rate, and i the rate at which the redemption fund can be accumulated, then the amount annually available for the redemption fund is

$$R - Vi'.$$

If this be accumulated for n years at rate i, and the fund then amounts to V, we have

$$(R - V \cdot i')s_{\overline{n}|} = V;$$

whence,

$$V = \frac{R}{\frac{1}{s_{\overline{n}|}} + i'} = \frac{R}{\frac{1}{a_{\overline{n}|}} + i' - i}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $s_{\overline{n}|}$ is to be computed at rate i. It is to be noted that this reduces to equation (11) when $i' = i$.

The foregoing results may also be obtained by using the sinking-fund method of estimating depreciation. For if V is the total capital to be invested in the property, including equipment, and the whole investment is to be abandoned in n years, and if D represents the annual payment into a sinking fund, sufficient to restore this capital at the end of n years, then, by (7),

$$D = V\frac{1}{s_{\overline{n}|}} \text{ (at rate } i\text{)}.$$

But the amount available for the sinking fund, after allowing for interest on the investment, was seen to be

$$R - V \cdot i'.$$

Hence,

$$V \frac{1}{s_{\overline{n|}}} = R - Vi'.$$

From which,

$$V = \frac{R}{\dfrac{1}{s_{\overline{n|}}} + i'}.$$

PROBLEMS

1. Find the value, on a 6 per cent basis, of a mine that can be made to yield a net annual income of \$20,000 for 15 years.

2. What will be the value of the mine in Problem 1, if the investment rate is 10 per cent, and the redemption fund rate is 5 per cent?

MISCELLANEOUS PROBLEMS

1. A man purchases a house for \$15,000. He pays \$6000 down and arranges to pay the balance, with interest at 7 per cent, in 8 equal annual installments. Find the annual payment. How much will still be due at the end of the fourth year, after his annual payment has been made?

2. Suppose that, in the preceding problem, the man accumulates a sinking fund to meet the payment of \$9000 at the end of the 8 years. If equal annual payments are made into the sinking fund, and it earns 4 per cent effective, what will be the total annual payment necessary to take care of both interest and sinking fund?

3. A debt of \$12,000, bearing interest at 7 per cent, is being paid off, principal and interest, by annual payments of \$2000. After four of these have been made, the balance then due is to be paid in four additional equal annual payments. How large must they be?

4. A debt of \$50,000, with interest at 6 per cent, is to be paid off by four equal annual installments, P, followed by four equal annual installments of $2P$. Find the value of P.

5. For 3 years, \$1000 is paid annually into a sinking fund earning 5 per cent per annum; after which the annual payment is increased to \$1500, but the rate of interest drops to 4 per cent. If the payments are made at the end of each year, how much will be in the sinking fund at the end of 8 years?

6. Construct a schedule showing the retirement of 50 bonds of par value $1000, and bearing 5 per cent, payable semiannually. The bonds are to be taken up at interest payment dates extending over 4 years, the schedule to be arranged so that the semiannual payment for interest and principal shall be as nearly equal as possible.

7. An article costing $1800 has a probable life of 8 years, with a residual value of $200. Construct a schedule of book values for each year of its life, (*a*) by the *sinking-fund* method, interest at 5 per cent; (*b*) by the *straight-line* method, (*c*) by the *constant percentage* plan.

8. Find the composite life of a plant consisting of two parts; *A* having a life of 50 years and a wearing value of $60,000, *B* having a wearing value of $20,000 and a probable life of 10 years. Assume that money is worth 5 per cent.

9. What constant percentage must be written off each year, if an article costing $10,000 is to be reduced to $5000 in 5 years?

10. If 4 per cent per year is written off on the book value each year, for a plant costing $100,000, what should the valuation be at the end of 10 years?

11. If a machine has a probable life of 15 years, and, without allowing for depreciation, yields an average net return of 10 per cent, what rate of income does it produce if a sinking fund is set aside at 5 per cent to replace it when it is worn out?

Ans. 5.366 per cent.

12. An automobile has a probable life of 6 years, and a wearing value of $2000. What is the annual cost to the owner, if his annual charge for upkeep and running expenses is $500, and depreciation, on the sinking-fund plan, is allowed on a 4 per cent basis?

13. How much should be paid for a mine that could be made to yield $25,000 a year for 12 years, after which it would have to be abandoned, if the investment rate is to be 12 per cent, while a redemption fund can be accumulated at 5 per cent?

CHAPTER IV

BONDS

33. Description.—The ordinary bond is a promise to pay a definite sum on a specified date, and, in the meantime, to pay interest on this sum at regular intervals, at a stated rate. These intervals are usually half-yearly, but interest may be paid quarterly or annually, or in some other regular manner.

To facilitate the payment of interest, the bond usually has coupons attached to it. These are themselves individual promises to pay the amount of the interest due at the respective interest-payment dates, and are detached and presented for payment as they fall due.

The sum named in the bond is called the *face value*, or *par value*; sometimes it is referred to as the *denomination* of the bond. When a bond falls due, and the specified payment of principal, and outstanding interest is made, the bond is said to be *redeemed*. Bonds are usually made redeemable at par, but sometimes, to make them more attractive to the investor, they are made redeemable above par. The bond interest, however, is always computed on the par, or face value.

34. The investment rate.—If a bond, redeemable at par, is bought at par, the investor will realize a rate of interest on his investment equal to the bond rate. Bonds, however, are usually bought at prices that yield an *investment rate*, independent of the interest rate named in the bond. If it is larger, then the purchase price is less than the par value. On the other hand, if the investment rate is smaller than the bond rate the purchase price is greater than the par value of the bond.

A number of considerations enter into the determination of a proper investment rate for a given bond. Chief among them are the value of the security back of the bond, the number of years before the bond matures, the marketability of the bond, in case the holder wishes to sell at any time, and the prevailing interest rates and opportunities for investment.

Assuming a given investment rate the mathematical theory of bonds is primarily concerned with the determination of the cost of a bond paying a specified coupon interest and with a given number of years to run before it matures. A second question is the determination of the investment rate when a bond is bought at a certain price. The first of these problems is solved directly by the bond formula.

35. The bond formula.—The following notation will be used:

F = the face, or par value of the bond,
C = the redemption price (usually $C=F$),
r = the bond rate of interest,
n = the number of years before redemption,
p = the number of interest payments per year,
i = the effective investment rate of interest,
V_n = the value of the bond n years before redemption.

The value of a bond n years before redemption is made up of two parts, (1) the present value of the redemption price, (2) the present value of all the interest payments. The latter constitute an annuity whose annual payment is rF. The present value of such an annuity is $rFa_{\overline{n|}}^{(p)}$. Combining these two quantities, we have

$$V_n = Cv^n + rFa_{\overline{n|}}^{(p)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v^n and $a_{\overline{n|}}^{(p)}$ are to be computed at rate i.

If, instead of i, the nominal rate $j_{(p)}$ is given, the interest paid on the bond may be thought of as an annuity of $\frac{rF}{p}$ per period, running np periods, interest being compounded at rate $\frac{j_{(p)}}{p}$. Hence,

$$V_n = Cv^{np} + \frac{rF}{p} a_{\overline{np|}}, \quad \left(\text{at rate } \frac{j_{(p)}}{p}\right). \quad . \quad . \quad . \quad (2)$$

It is the practice of bond agencies to quote the price of a bond to yield a certain specified investment rate, meaning thereby the nominal rate corresponding to the number of interest payments per year. Thus, a bond sold at a price to yield $6\frac{1}{2}$ per cent to the investor means, when interest is payable semiannually, an investment rate of $3\frac{1}{4}$ per cent per half year ($j_{(2)}=0.065$). Tables are published giving prices of bonds with yearly, semiannual and quarterly interest payments. The investment rates, as quoted in these tables, are nominal rates, compounding in agreement with interest periods. Unless otherwise indicated, this procedure will be followed in this chapter.

It is also the practice to quote bond prices on the basis of 100 par value. Thus, a quotation of 98.42 means that a $1000 bond at that price would sell for $984.20.

Example. (1) Find the cost of a 15-year 5 per cent bond, redeemable at par, interest payable semiannually, bought at a price to yield 6 per cent.

Here, $j_{(2)}=0.06$ and $r=0.05$. The semiannual interest payments on $100 par value are $2.50. Hence, from (2)

$$V_{15}=100v^{30}+2\tfrac{1}{2}a_{\overline{30|}} \quad \text{(at 3 per cent).}$$

From the tables,

$$100v^{30}=41.20,$$

$$2\tfrac{1}{2}a_{\overline{30|}}=49.00.$$

Hence,

$$V_{15}=90.20.$$

(2) Find the cost of the bond in Example (1) if redeemed at 105.

In this case,

$$V_{15}=105v^{30}+2\tfrac{1}{2}a_{\overline{30|}}, \quad \text{(at 3 per cent)}$$

$$105v^{30}=43.26,$$

$$2\tfrac{1}{2}a_{\overline{30|}}=49.00.$$

Hence,

$$V_{15}=92.26.$$

Unless otherwise stated, it will be understood that bonds are redeemable at par.

PROBLEMS

1. Find the cost of a 4 per cent $1000 bond, redeemable in 40 years, interest payable semiannually, bought at a price to yield 5 per cent.
Ans. $827.74.

2. Find the cost of a 5 per cent $1000 bond, redeemable in 40 years, interest payable semiannually, bought at a price to yield 4 per cent.
Ans. $1198.72.

3. At what price should 5-year 6 per cent bonds be quoted, if redeemable at 105, interest payable semiannually, if they are to yield 8 per cent?
Ans. 95.27.

4. Compare the costs of two 5 per cent bonds, each paying interest semiannually, one maturing in 5 years, the other in 10 years, bought to yield $4\frac{1}{2}$ per cent. *Ans.* 102.22; 103.99.

5. Find the cost of a 6 per cent bond, par value $1000, interest payable quarterly, bought at a price to yield 7 per cent, maturing in 10 years
Ans. $928.51.

6. Compare the costs of $1000 bonds, paying 6 per cent interest, redeemable in 10 years, bought at prices to yield the investor 5 per cent, one paying interest annually, a second semiannually, and a third quarterly.

7. Find the cost of a $1000 bond, redeemable in 10 years at 106, paying 6 per cent quarterly, bought at a price to yield 7 per cent.

8. At what price should 20-year, 5 per cent semiannual bonds be quoted to yield the purchaser $4\frac{3}{4}$ per cent?

36. Premium and discount.—Further insight into the valuation of bonds may be obtained by calculating the difference between the price paid for a bond and its face value. When this difference is positive it is called the *premium*; when negative it is spoken of as the *discount*.

To simplify the discussion, it will be assumed that the bond is redeemed at par. Letting $C=F$ in (1) we have

$$V_n - F = F(v^n - 1) + rFa_{\overline{n}|}^{(p)}.$$

But, from (5) § 16,

$$v^n - 1 = -j_{(p)} \cdot a_{\overline{n}|}^{(p)}.$$

Hence,

$$V_n - F = (r - j_{(p)})Fa_{\overline{n}|}^{(p)}. \qquad (3)$$

If interest is considered as compounded in agreement with the coupon payments on the bond, then, from (4) § 16

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} a_{\overline{np}|}. \quad \left(\text{at rate } \frac{j_{(p)}}{p}\right).$$

Hence (3) becomes

$$V_n - F = \left(\frac{r}{p} - \frac{j_{(p)}}{p}\right) F \cdot a_{\overline{np}|}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These results show, and it is otherwise obvious, that a bond sells at a premium when $r > j_{(p)}$, and at a discount when $r < j_{(p)}$. The premium is seen from (3), or (4) to be equal to the present value of an annuity, running n years, whose periodic payment is equal to the difference between the interest paid on the bond and that required by the investment yield. This result is apparent also by direct reasoning, because the cost would equal F if the yield rate were the same as the investment rate. The excess of the bond interest over the yield requirement, therefore, constitutes an additional periodic income, or annuity, whose present value the purchaser pays for in the form of the premium.

In the case when the bond interest is less than the yield demand, the purchaser may be regarded as having paid F for the bond; but the seller gives him a rebate, or discount equal to the present value of the deficiency in the coupon payments from the amounts that would have been required had the purchase price been F.

Since the cost of an annuity is greater the longer it has to run, so the premium or discount will be greater the longer the life of the bond, the other data being the same.

PROBLEMS

1. Compute the premium on a \$1000 bond, paying 6 per cent, with 20 years to run, interest payable semiannually, bought at a price to yield per cent. What would the premium be if the life of this same bond was 40 years?

2. Find the discount on a \$100, 4 per cent bond, bought 10 years before maturity, interest payable quarterly, yielding 6 per cent to the investor. What would the discount be one year before maturity?

3. An issue of 6 per cent bonds, maturing in 20 years, interest payable semiannually is bought at a price to net $5\frac{3}{4}$ per cent. Find the premium paid.

37. Amortization of the premium.—When a bond is bought at a premium, care should be taken by the investor, to conserve all of the capital involved in the purchase. The capital represented by the face value of the bond is returned to him when the bond matures. The capital invested in the premium, however, is returned as part of the excess of interest over yield requirements, as seen in the previous article. The amortization of the premium follows the same process, as illustrated in § 24. At each interest payment, a portion of the premium is returned, and the book value of the bond is "written down" by that amount, gradually approaching par at maturity.

As an illustration, consider a 6 per cent \$1000 bond, redeemable at par, January 1, 1923, interest payable on January 1 and July 1, bought January 1, 1920, at a price to yield 5 per cent. The premium is found to be \$27.54. From (4), this is the present value of an annuity whose semiannual payment is

$$\frac{1}{p}(r - j_{(p)})F = \$5.00.$$

Proceeding as in § 24, it is found that the interest on \$27.54 for 6 months at 5 per cent is \$0.69. Hence, on July 1, 1920, the value of the premium has been reduced by \$4.31. The book value of the bond is, therefore, \$1023.23. During the next 6 months, the interest on the remainder of the premium is \$0.58; on January 1, 1921, therefore, \$4.42 of capital is returned, leaving the book value of the bond as \$1018.81. This process continues until the date of maturity, at which time the entire premium has been returned and the value of the bond stands at par.

The process of accounting will be found simpler if the premium be not segregated from the rest of the capital invested. Thus, the accountant would charge $1027.54 against this bond investment, as of date January 1, 1920. On July 1, 1920, a $30 coupon is cashed. The investment at 5 per cent requires, however, only $25.69; hence, $4.31 remains for amortization of the premium. The continuation of the accounting is shown in the following table:

Date.	Book Value.	Semiannual Interest at $2\frac{1}{2}$ Per Cent on Book Value.	Semiannual Bond Interest at 3 Per Cent on Par.	For Amortization.
Jan. 1, 1920....	$1027.54			
July 1	1023.23	$25.69	$ 30	$4.31
Jan. 1, 1921....	1018.81	25.58	30	4.42
July 1	1014.28	25.47	30	4.53
Jan. 1, 1922....	1009.64	25.36	30	4.64
July 1	1004.88	25.24	30	4.76
Jan. 1, 1923....	1000.00	25.12	30	4.88
		$152.46	$180	$27.54

An examination of this table shows that, of the $180 received through interest payments, only $152.46 may be regarded as income on the investment. The sums in the last column, to the amount of $27.54, are return of capital represented by the premium. These amounts, as they come in, should be returned to the capital account of the owner of the bond, to be again invested, but such investment would be separated from, and independent of, the bond investment here considered.

It should be noted, however, that had the owner of the bond reinvested $4.31 each half year at 5 per cent, he could have left the investment on his books at the original cost, $1027.54, and regarded the $25.69 as a uniform income from the bond.

The accumulation of the annuity thus created by the semi-annual investment of \$4.31 would amount to

$$\$4.31\ s_{\overline{6}|} = \$27.54,$$

at the date when the bond matured. This amount, together with the return of the face value of the bond, would preserve the total amount of capital intact.

PROBLEMS

1. Construct the schedule showing the amortization of the premium on a \$10,000 bond, bearing 5 per cent interest payable January 1 and July 1, redeemable at par, January 1, 1926, bought July 1, 1922, at a price to yield 4 per cent.

2. Construct a schedule for a \$1000 bond, bearing 8 per cent interest payable quarterly, redeemable in two years at 105, bought at a price to yield 6 per cent to the investor. This schedule will show the amortization of the excess of the purchase price over \$1050, the redemption price.

38. Accumulation of discount. When a bond is bought at a price which is less than the redemption price, its value increases as the date of maturity approaches. The bond rate of interest is not sufficient, in this case to give the income required by the investment. The deficit is the amount by which the book value of the bond increases at each interest-payment date. This process of "writing up" the book value is also called *accumulating the discount.* This phrase does not conform to the definition of discount, except when the bond is redeemed at par.

Example.—Construct a schedule showing the accumulation of the discount on a \$1000 bond, bearing 4 per cent interest, payable January 1 and July 1, bought January 1, 1920, at a price to yield 5 per cent, the bond redeemable at par January 1, 1923.

The purchase price is found to be \$972.46. Interest on this amount for 6 months, at the investment rate, is \$24.31. The coupon, however, pays only \$20. The value of the bond, as computed for this date, would be \$976.77, which is \$4.31 greater than its value 6 months earlier. This is just the amount represented by the difference between the yield requirement and the bond interest. The schedule shows the continuation of this process to maturity.

Date.	Book Value.	Semiannual Interest at $2\frac{1}{2}$ Per Cent on Book Value.	Semiannual Bond Interest at 2 Per Cent Cent on Par.	Accumulation of Discount.
Jan. 1, 1920	$972.46			
July 1	976.77	$24.31	$ 20	$4.31
Jan. 1, 1921	981.19	24.42	20	4.42
July 1	985.72	24.53	20	4.53
Jan. 1, 1922	990.36	24.64	20	4.64
July 1	995.12	24.76	20	4.76
Jan. 1, 1923	1000.00	24.88	20	4.88
		$147.54	$120	$27.54

PROBLEMS

1. Construct the schedule showing the accumulation of the discount on a $1000 bond bearing 5 per cent interest, payable March 1 and September 1, redeemable at par September 1, 1928, bought March 1, 1923, at a price to yield 6 per cent.

2. Construct the schedule for a $1000 bond, redeemable in 2 years at 102, bearing interest at 6 per cent, payable quarterly, bought at a price to yield 8 per cent. (This schedule will show the increase in the book value at interest dates, finally reaching $1020 at maturity.)

39. Bonds purchased between interest-payment dates.—When a bond is purchased between interest-payment dates, the seller is clearly entitled to a portion of the interest that has accrued since the last coupon was paid. If V is the value of the bond at that date, the *theoretical* value of the bond would be V, together with interest on this amount, at the investment rate, for the portion of the period that has elapsed.

Example.—Find the value of a $1000 bond bearing 5 per cent, interest payable semiannually, maturing at par January 1, 1927, bought May 1, 1922, at a price to yield $4\frac{1}{2}$ per cent. On January 1, 1922, its value was $1022.17. The accumulated amount for 4 months at $4\frac{1}{2}$ per cent on this sum would be

$$\$1022.17\ (1.0225)^{2/3} = \$1037.45.$$

In practice, however, this amount would be computed by simple interest.

It is the practice of bond houses and other selling agencies to quote bonds at a given price *with accrued interest*, rather than a price computed on a strictly yield basis. This means that simple interest at the rate named in the bond is computed for whatever fraction of a period may have elapsed since the last interest was paid. Thus, in the foregoing example, if the quoted price was \$1022.17 and accrued interest, the accrued interest for 4 months at 5 per cent on the face of the bond, would be \$16.67, making the purchase price \$1038.84.

In consideration of the fact that the next interest payment is 2 months hence, the buyer would be entitled to a discount, amounting to \$0.14, for advancing the \$16.67. This discount is negligible for small transactions, but may become quite appreciable when large sums are involved.

PROBLEMS

1. A \$100 bond maturing October 1, 1930, bearing 6 per cent interest payable April 1 and October 1, is bought August 1, 1923, at a price to yield 5 per cent to the investor. Find the theoretical value of the bond.

2. A 5 per cent bond, par value \$10,000, maturing January 1, 1927, interest payable January 1 and July 1, is sold June 1, 1923, on a 6 per cent yield basis. Find its selling price. What would its selling price be if it were sold on the basis of its value on January 1, 1922, plus accrued interest?

3. A \$1000 bond, paying 7 per cent semiannually, on April 1 and October 1, was sold on March 1, at 102.25 and accrued interest. What was the selling price?

4. Find the theoretical value of a \$100 bond bearing 5 per cent interest, payable July 1 and January 1, maturing January 1, 1925, bought June 1, 1923, on a 6 per cent yield basis.

40. Calculation of the investment rate when the purchase price is given.—The prices of bonds in the open market are subject to fluctuations. A purchaser, in determining whether a particular bond at the quoted market price offers the kind of investment he desires, should know what rate of interest the bond will yield. Its determination, then, is of great practical importance; but, as is the case with many inverse problems, its accurate calculation offers mathematical difficulties.

An examination of the bond formula, in the form given by (1),

$$V_n = Cv_n + rFa_{\overline{n|}},$$

shows that the yield rate, i, enters into v and into $a_{\overline{n}|}$, and when V_n is given, this equation, expressed in terms of i, is of degree $n+1$. When the value of i, to several decimal places, is desired, methods of solution developed in the theory of equations may be applied. For practical purposes, however, it is possible to determine the approximate yield rate by simple interpolation. In order to illustrate this method it will first be assumed that only such tables as are given in this book are available.

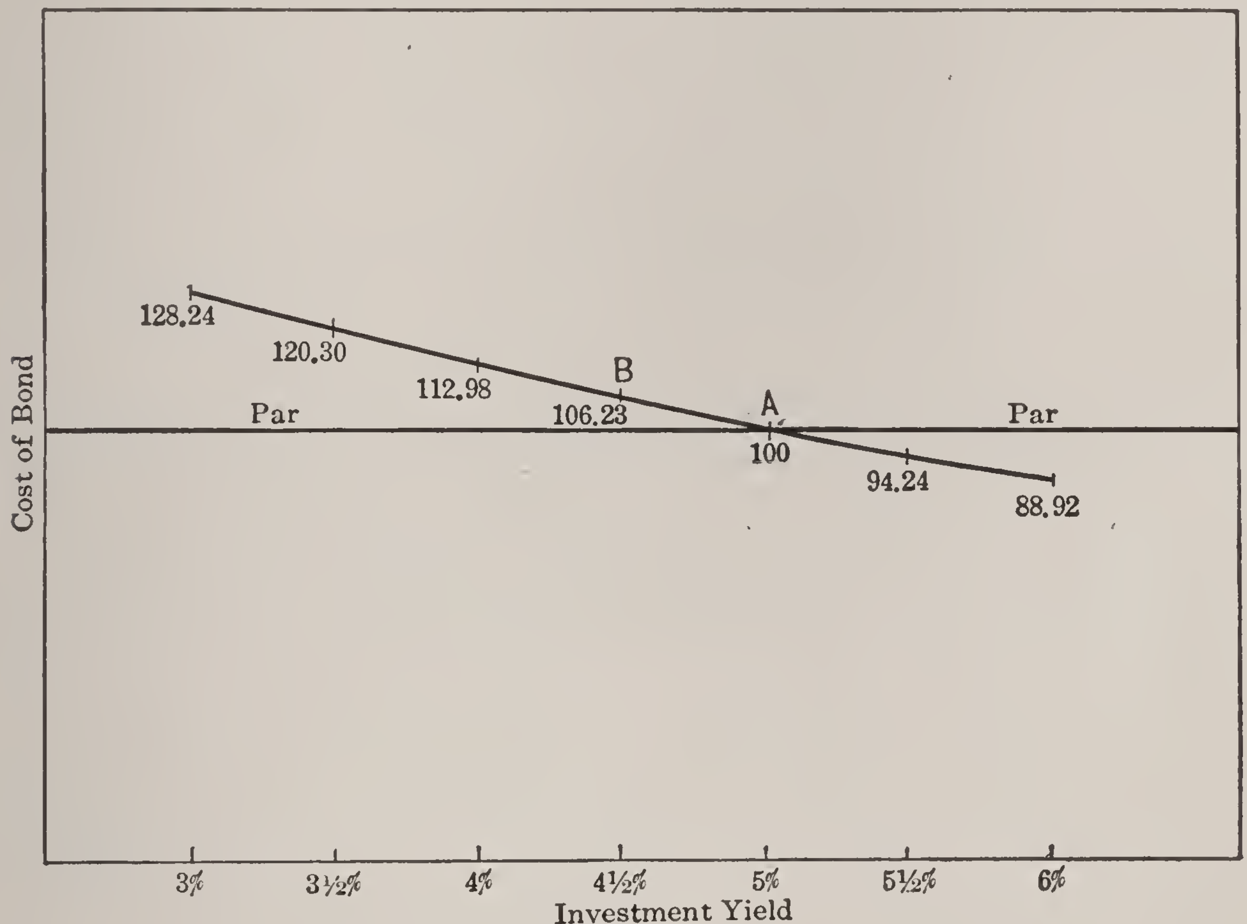

FIG. 3.

Consider, as an example, a 5 per cent bond maturing in $18\frac{1}{2}$ years, interest payable semiannually, redeemable at par. Figure 3 shows the manner in which the cost diminishes as the investment rate varies from 3 per cent to 6 per cent, the cost being computed for each .5 per cent interval. The curve obtained by joining the points representing the costs is seen to be slightly concave upward. Between two adjacent points, such as A and B, the curve may be assumed to be a straight line, and the yield corresponding to a price falling between the prices represented by A and B may be determined by interpolation. Thus, suppose the quoted price to be 103.35 and the resulting yield x.

Cost.	*Yield.*
106.23	$4\frac{1}{2}$ per cent
103.35	x per cent
100.00	5 per cent

Hence,

$$2.88 : 6.23 = x - 4\tfrac{1}{2} : \tfrac{1}{2}.$$

From which one obtains $x = 4.73$ per cent.

This method will always give a result slightly larger than the correct answer, because the chord is above the curve, at the point where the cost is plotted. The correct answer, in the foregoing example, to three decimals, is 4.726 per cent.

It is obvious that the shorter the chord used the better will be the result obtained. The procedure is to estimate about where the yield will be and to compute the cost at two rates nearest to this estimate, one smaller and the other larger, these rates to be determined by rates of interest used in the available annuity tables.

Bond tables may be obtained which give investment rates differing by $\frac{1}{20}$ of 1 per cent. It is therefore possible by the use of these tables to obtain results of still greater accuracy. Thus, in the preceding example, the bond costs nearest to 103.35 are

103.05, corresponding to a yield of 4.75 per cent.
103.68, corresponding to a yield of 4.70 per cent.

Hence, if 103.35 corresponds to a yield of x per cent, then,

$$0.33 : 0.63 = x - 4.70 : 0.05.$$

Hence,

$$x = 4.7262.$$

MISCELLANEOUS PROBLEMS

1. Three \$1000 bonds, paying respectively 4 per cent, 5 per cent, and 6 per cent semiannually, and all maturing in 15 years, are bought at prices to yield 7 per cent. Compare the prices and show that they form an arithmetical progression.

Ans. \$724.12; \$816.08; \$908.04.

2. Prove that the prices of any three bonds of like denomination are in arithmetical progression, provided they have the same yield rate and date of maturity, and bear interest rates that are in arithmetical progression.

3. From the results of Problem 2, find the value of an 8 per cent bond, assuming that a 6 per cent bond of the same denomination, date of maturity, and yield rate, is worth \$98.75, while a similar 7 per cent bond is worth \$101.25.

4. A corporation issues \$100,000 in 6 per cent bonds, redeemable at par in 15 years, interest payable semiannually. They accumulate a sinking fund at 4 pe· cent, converted semiannually. Find the semiannual payment necessary to meet both interest and sinking-fund charges. Conside ing this semiannual payment as an annuity, and assuming that the bonds sold at 98, find the approximate rate of interest, that the corporation has to pay for the use of this money.

5. Suppose, in Problem 4 that 20-year bonds had been issued paying 7 per cent, semiannually, and that the issue on this basis would sell at 102; with the same provision for a sinking fund, which plan would be more advantageous to the corporation?

6. A corporation issues 5-year bonds, bearing 6 per cent interest payable semiannually. If they sell at 96.50, what rate of income do they yield?

7. A \$10,000 bond, due July 1, 1930, is sold on April 15, 1923, at a price to yield the purchaser 7 per cent. The bond bears 6 per cent, interest payable January 1 and July 1. Find its *theoretical* value, also its selling price, based on its value January 1, 1923, plus accrued interest.

8. Find the rate of income realized on a 7 per cent bond purchased for 102.50, paying interest semiannually, and bought 20 years before maturity

9. An 8 per cent bond, redeemable at 103, interest payable quarterly, is bought at 98, 5 years before maturity. Find the rate of interest realized by the purchaser.

CHAPTER V

PROBABILITY

41. Definition of probability.—In the toss of a coin, the chances of its coming "heads" are $\frac{1}{2}$, as are also the chances of its coming "tails;" in the throw of a die, the chances of a particular face coming upward are $\frac{1}{6}$. With the coin, there are two possibilities, each equally likely, and, in the long run, half of the throws may be expected to be "heads" and the other half "tails." With the die, one would expect that, of a great number of throws, one-sixth would bring a particular face upward.

This ratio of the number of favorable ways in which an event may happen to the total number of ways, favorable and unfavorable, is used as the mathematical measure of probability. The definition may be algebraically stated as follows:

If an event can happen in m ways, r of which are favorable and s unfavorable, then the probability of a favorable occurrence is $\frac{r}{m}$, and the probability of an unfavorable occurrence is $\frac{s}{m}$, where $r+s=m$.

It follows that,

$$\frac{r}{m}+\frac{s}{m}=1;$$

hence, if $p=\frac{r}{m}$, then $1-p=\frac{s}{m}$; therefore, if the probability of an event happening is p, then the probability of its not happening is $1-p$.

Again, if an event is *certain*, it will happen without fail in every case; therefore 1 is the mathematical measure of *certainty*.

To illustrate further: If a bag contains 7 balls, 3 of which are white and 4 black, then there are 7 ways of drawing a ball from the bag, in 3 of which the ball is white, while in 4 it is black. The probability of drawing a white ball is then $\frac{3}{7}$, and that of drawing a black ball $\frac{4}{7}$.

While it is possible in many cases, as in the foregoing illustration, to enumerate accurately the total number of possible ways in which an event may occur favorably or unfavorably, there are other types in which probable future occurrences must be based on statistical data obtained through experience. Thus, if it has been observed that, of a certain class of buildings under similar conditions, one out of every n has been lost through fire each year, then $\frac{1}{n}$ may be taken as the probability that a particular building of this type will burn in any given year.

The American Experience Table of Mortality (Table X), shows that, of 100,000 males aged 10, there will be 57,917 living at age 60. Hence, the probability that a particular individual of the original group will attain the age of 60 is

$$\frac{57{,}917}{100{,}000}=0.57917.$$

It should be noted, throughout this discussion, that the occurrence of an event, and the occurrence of any other event with which the given event is compared, are assumed to be "equally likely." Thus, in the toss of a coin, or the throwing of a die, any particular face is as likely to come up as any other. On the other hand, in the problem just considered, it would be erroneous to assume that dying before the age of 60, or surviving that age, are equally likely events, for, if they were, the probability of each would be $\frac{1}{2}$. Put another way, we assume that in the long run, or with a large number of cases, or trials, the events under consideration will occur equally often.

Before applying the theory of probability to questions of insurance, it is necessary to develop some fundamental theorems.

42. Theorems on arrangement and combination.—(i) If an act can be performed in a ways and, after it has been completed in one of these ways, another act can be performed in b ways, then the two acts can be performed, in the order stated, in ab ways.

The truth of this statement will be obvious, without formal proof. As an illustration, suppose that there are 3 routes from a city A to a city B, and that there are 5 ways of traveling from B to a third city C; then it is clear that one can go from A to C in 15 ways, two routes being considered different if they differ in any part of the journey.

(ii) *Permutations.*—If, from a group of n things, sets of r are to be chosen and arranged in order, each arrangement is called a *permutation.* Two permutations are said to be different if they do not contain the same r elements throughout, or if the same r elements appear in both, but occur in a different order. The arrangement *abcd* differs from the arrangement *abce*, and, with the same letters, the permutation *abc* is different from *acb*.

Indicating the n elements, or objects, by the letters a_1, a_2, ..., a_n, we may choose the first one in n ways. After one letter has been selected, the second one can be chosen in $n-1$ ways. Hence, by (i), the first two places can be filled in $n(n-1)$ ways. Proceeding in this manner, we find that the first three places can be filled in $n(n-1)(n-2)$ ways, etc. Denoting the total number of ways of filling the r places by ${}_nP_r$, we have

$$ {}_nP_r = n(n-1)(n-2) \ldots . (n-r+1). \quad . \quad . \quad (1) $$

In particular, if $r=n$,

$$ {}_nP_n = n(n-1)(n-2) \ldots 3\cdot 2\cdot 1 = n! \quad . \quad . \quad . \quad (2) $$

Thus,

$$ {}_5P_3 = 5\cdot 4\cdot 3 = 60, $$

while

$$ {}_5P_5 = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5! = 120. $$

(iii) *Combinations.*—When r things are to be chosen from a group of n things, *without regard to the order of arrangement* in any selection, the group of r objects chosen is called a *combination* of the n things into a set of r. The total number of combinations into sets, each containing r objects, that it is possible to make from the set of n, is indicated by the symbol ${}_nC_r$. Two combinations are regarded as different if one of them possesses an element not found in the other.

To obtain the value of ${}_nC_r$, one needs only to note that, by permuting all of the r elements in a particular combination, $r!$ permutations may be obtained. Doing this in every one of the ${}_nC_r$ combinations, we obtain a total of ${}_nC_r \cdot r!$ permutations. But every possible permutation is present in this total, since each permutation is present in some combination; hence,

$$ {}_nC_r \cdot r! = {}_nP_r, $$

or

$$ {}_nC_r = \frac{{}_nP_r}{r!}. \qquad (3) $$

Corollary.—Every time a choice of r things is made, $n-r$ things are left behind; hence,

$$ {}_nC_r = {}_nC_{n-r}. \qquad (4) $$

(iv). The expansion for $(a+b)^n$ is given by the formula,

$$ (a+b)^n = a^n + {}_nC_1 a^{n-1} b + {}_nC_2 a^{n-2} b^2 + \ldots + {}_nC_r a^{n-r} b^r + \ldots + b^n, \qquad (5) $$

for n a positive integer.

The truth of this theorem may be seen by considering $a+b$ multiplied by itself n times. In the continued product, one forms every possible term by taking one letter from each of the n factors. Thus, the term $a^{n-r}b^r$ can be formed in as many ways as the letter b can be selected r times from the n factors, the letter a being chosen from the remaining $n-r$ factors. The total number of such terms in the product will therefore be ${}_nC_r$.

PROBLEMS

1. Write down the values of the following symbols: ${}_4C_3$, ${}_6P_2$, ${}_nP_2$, ${}_nC_2$.

2. In how many ways can 5 people be arranged in a row?

3. How many matches of tennis singles will have to be played by 10 players if each one is to play against every other player?

4. Prove algebraically that ${}_nC_r = {}_nC_{n-r}$.

5. In how many ways can 50 cards be chosen from a pack of 52?

6. Eight people are arranged at random in a circle. What is the probability that a particular pair will be together?

7. From a bag containing 5 white balls and 6 black balls, 2 are drawn at random. What is the probability that both will be white? Both black? One white and one black? What is the sum of the three results?

8. Expand $(a+b)^6$ using formula (5).

9. Five coins are tossed; what is the probability that exactly three of them are "heads"?

43. Mutually exclusive events.—Two or more events are said to be *mutually exclusive* when the occurrence of any one of them precludes the possibility of any of the others.

As an illustration, suppose that a single ball is to be drawn from a bag containing balls of different colors. The drawing of a ball of one color precludes the possibility of drawing one of another color. Thus, if there are white and red balls in the bag, the drawing of a white ball and the drawing of a red ball are mutually exclusive events.

In the foregoing illustration, suppose that there are, in all 13 balls in the bag, 2 of which are white and 5 red. Then the probability of drawing a white ball is $\frac{2}{13}$, and of drawing a red ball is $\frac{5}{13}$. The number of balls that are either red or white is 7. Hence the probability of drawing *either* a red or a white ball is $\frac{7}{13}$, or the *sum* of the probability of drawing a red ball, and of the probability of drawing a white ball.

From the definition of mutually exclusive events, it follows, then, that *the probability of one or the other of them happening is the sum of their separate probabilities.*

44. Compound events.—Events are said to be *independent* when the occurrence of any one of them does not affect, in any way, the occurrence of any of the others. Thus, if two coins be tossed, the result for one of them is independent of the throw of the other. The survival of one person for a given number of years does not affect the probability that a second person will also be alive at the end of that period. These would be called independent events.

When two coins are tossed, 4 different results may occur. Of these, for example, one result will be two heads. The probability, therefore, of throwing two heads with two coins is $\frac{1}{4}$, which is the *product* of the probabilities that *each* will register heads.

Again, if a coin and a die be tossed, the probability that a head and an ace will come up is $\frac{1}{12}$, because there are 12 possible results, only one of which consists of a head and an ace. But the probability that a head will come is $\frac{1}{2}$, and that an ace will be thrown is $\frac{1}{6}$. The probability, then that both of these independent events will occur is the product of their separate probabilities.

The truth of this principle will be established for the joint occurrence of any two independent events.

Suppose that the first can happen in a_1 ways and fail in b_1 ways, while the second can happen in a_2 ways and fail in b_2 ways. Each of the a_1+b_1 possible results of the first event can be associated with each of the a_2+b_2 results of the second, making a total of

$$(a_1+b_1)\,(a_2+b_2)=a_1a_2+a_1b_2+a_2b_1+b_1b_2$$

possible cases of joint occurrence. In a_1a_2 of these, both events occur. Hence, the probability that both events will happen is

$$\frac{a_1a_2}{(a_1+b_1)(a_2+b_2)}=\frac{a_1}{a_1+b_1}\cdot\frac{a_2}{a_2+b_2}.$$

Similarly, if p_1, p_2, . . ., p_n are the respective probabilities that n independent events will happen, then the probability that they will all occur is $p_1p_2p_3 \ldots p_n$; while the probability that they will all fail is

$$(1-p_1)(1-p_2) \ldots (1-p_n).$$

PROBLEMS

1. What is the chance of throwing a 1 or a 6, in a single throw of a die?

2. What is the probability of throwing a 1 followed by a 6, in 2 throws of a die?

3. What is the probability of throwing 3 heads in 3 throws of a coin?

4. Find the probability of not throwing a 6 in 3 throws of a coin.

5. If the probability of A living a certain time is $\frac{4}{5}$, and that of B surviving for the same period is $\frac{5}{6}$, what is the probability that A will die and B will survive? That both will die?

45. Probability with repeated trials.—If the probability of an event happening in one trial is p, the probability of its failing is $q=1-p$. Hence, the probability of its happening r times in n trials and failing $n-r$ times, *in any specified order*, is, from § 44, equal to p^rq^{n-r}. But the number of ways one can designate a particular order in which the event is to happen and fail is the number of ways one can choose r numbers from a set of n, or ${}_nC_r$, and these ways are all equally probable and mutually exclusive. Hence, the probability of the event happening exactly r times in n trials is ${}_nC_rp^rq^{n-r}$.

Example.—Suppose the probability of an event happening in one trial is $\frac{1}{6}$. Find the probability that it will happen 3 times in 5 trials. There are ${}_5C_3=10$ ways one can designate the 3 trials from among the 5 in which the event is to occur, and in the other two to fail. Thus, it might happen in the first, third and fourth, and fail in the second and fifth. The probability that the results shall be in this specified order is

$$\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}=\frac{1}{6^3}\cdot\left(\frac{5}{6}\right)^2.$$

But there are 10 such orders possible, in each of which the event would happen exactly 3 times. Hence, the probability of the event happening 3 times in 5 trials is

$$10\cdot\frac{1}{6^3}\cdot\left(\frac{5}{6}\right)^2=\frac{125}{3888}.$$

The probability of an event happening *at least* r times in n trials is the sum of the probabilities of its happening exactly r, $r+1$, $r+2$, . . . up to n times, or

$${}_nC_r p^r q^{n-r}+{}_nC_{r+1}p^{r+1}q^{n-r-1}+\ \dots\ +p^n.$$

In the present example, the probability of the event happening *at least* 3 times in 5 trials is then

$${}_5C_3\frac{1}{6^3}\left(\frac{5}{6}\right)^2+{}_5C_4\frac{1}{6^4}\left(\frac{5}{6}\right)+\frac{1}{6^5}=\frac{23}{648}.$$

It should be noticed that, by (iv) § 42, since $p+p=1$, we have

$$(q+p)^n=1=q^n+{}_nC_1pq^{n-1}+{}_nC_2p^2q^{n-2}+\dots+{}_nC_rp^rq^{n-r}+\dots+p^n.$$

The truth of this becomes obvious if one notes that the right side is the sum of the probabilities that the event will fail every time, happen exactly once, or twice, etc., up to and including the probability that it will happen every time. One of these must happen; hence, the sum of their probabilities is 1, the mathematical measure of certainty. The most probable number of successes and failures would be given by the greatest term in this expansion.

PROBLEMS

1. Find the probability of throwing exactly 2 heads in 6 throws of a coin. At least 2 heads. *Ans.* $\frac{15}{64}$; $\frac{57}{64}$.

2. If the chances of a team winning a particular game are $\frac{1}{3}$, what is the probability that it will win exactly 2 games out of a series of 3? At least 2 games? *Ans.* $\frac{54}{125}$; $\frac{81}{125}$.

3. In a World's Series baseball contest, the teams were rated even. What was the probability that one team would win 4 out of the first 5 games played and lose the other one. *Ans.* $\frac{5}{32}$.

4. If, on a certain coast, one steamer is lost out of every 250 trips undertaken, what is the probability that of 10 expected to arrive at least one will be lost?

5. If the probability of an event happening in one trial is $\frac{1}{3}$, what is the most probable number of favorable occurrences in 6 trials?

Ans. 2.

46. Mathematical expectation.—If p is the probability of obtaining a sum of money, M, then pM is the value of the *expectation.* If in a large number, m, of cases or trials the sum M is received a times, the average amount received for each trial is $\frac{aM}{m}$. But, $p=\frac{a}{m}$; therefore, the expectation may be considered as the average amount received *in the long run* for each trial.

If a person holds one ticket in a lottery containing 100 tickets and having one prize of \$25, then, in the long run, he may expect to win the prize once out of every 100 drawings. He would, therefore, pay \$0.25 for each chance, or $\frac{1}{100}$ of the prize. His expectation, then, is valued at \$0.25.

As a further illustration, suppose that 1000 men, all aged 35, contribute to a fund, with the understanding that each survivor will receive \$100 at the end of 10 years. The mortality tables show that approximately 907 will be alive. Hence, the expectation of each would be valued at \$90.70. The fund will have to contain \$90,700, if each survivor is to receive \$100. Hence, neglecting interest, each of the 1000 men will have to contribute \$90.70.

47. Mortality tables.—Through the experience of insurance companies and other agencies, tables have been constructed that indicate the number of people, from a given initial group, that die in each succeeding year. These data are usually expressed in terms of some convenient initial number, as in Table X, where the number of persons living at age 10 is taken as 100,000.

The number of persons living at the age x is denoted by l_x, and the number dying in the age interval from x to $x+1$ by d_x. The table shows the values of l_x and of d_x for every year from $x=10$ to $x=95$. Three are assumed to be alive at the latter age, but to die during the year. The symbol (x) is used to indicate a person aged x. The probability that (x) will live at least one year is denoted by p_x, and that he will die within the year by q_x. The following relations exist between these quantities:

$$d_x = l_x - l_{x+1}. \qquad (6)$$

$$p_x = \frac{l_{x+1}}{l_x}, \qquad (7)$$

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = 1 - p_x. \qquad (8)$$

From the nature of these quantities, it is clear that

$$l_x = d_x + d_{x+1} + d_{x+2} \ldots \text{to end of table.} \qquad (9)$$

$$l_x - l_{x+n} = d_x + d_{x+1} + d_{x+2} + \ldots + d_{x+n-1}. \qquad (10)$$

The probability that (x) will live at least n years is denoted by ${}_np_x$; hence,

$${}_np_x = \frac{l_{x+n}}{l_x}. \qquad (11)$$

The probability that (x) will not survive n years is denoted by ${}_{|n}q_x$. It follows that

$${}_{|n}q_x = 1 - {}_np_x = \frac{l_x - l_{x+n}}{l_x}. \qquad (12)$$

48. Joint life probabilities.—The survival of (x) and the survival of (y) are independent events. If ${}_np_{xy}$ denote the probability that both will live n years, we have

$${}_np_{xy} = {}_np_x \cdot {}_np_y. \qquad (13)$$

The probability that (x) will live n years, and (y) will die within that period is

$${}_np_x \cdot {}_{|n}q_y = {}_np_x(1 - {}_np_y). \qquad (14)$$

The probability that at least one of two lives (x) and (y) will survive n years is the sum of the probabilities that both will live, and that either (x) or (y) will survive and the other die.

Example 1.—Find the probability that a man aged 25 will be alive 40 years later:

By formula (11)

$$ {}_{40}p_{25} = \frac{l_{65}}{l_{25}}. $$

From Table X,

$$ l_{65} = 49{,}341 \text{ and } l_{25} = 89{,}032; $$

hence,

$$ {}_{40}p_{25} = \frac{49{,}341}{89{,}032} = 0.55419. $$

Example 2.—Find the probability that A, aged 30, will live 10 years and B, aged 25, will die within that period.

By formula (14), the desired probability is given by

$$ {}_{10}p_{30}(1 - {}_{10}p_{25}). $$

From Table X,

$$ {}_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{78{,}106}{85{,}441} = 0.91415, $$

$$ {}_{10}p_{25} = \frac{l_{35}}{l_{25}} = \frac{81{,}822}{89{,}032} = 0.91902, $$

$$ 1 - {}_{10}p_{25} = 0.08098. $$

Hence,

$$ {}_{10}p_{30}(1 - {}_{10}p_{25}) = 0.07403. $$

PROBLEMS

1. Find the probability that a man aged 45 will live to be 65.

2. Find the probability that a man aged 70 will die within one year. Within 10 years.

3. Calculate the probability that A, aged 35, and B, aged 40, will both survive 10 years. What is the probability that at least one of them will survive?

4. What is the probability that three persons, each aged 21, will all reach the age of 60? What is the probability that none will reach that age?

5. What is the probability that three persons, A, B, and C, all of the same age, will die in any given order, such as A, B, C?

6. A boy aged 15 is to receive \$10,000 on his twenty-first birthday. What is the value of his expectation on a 5 per cent basis?

7. Prove that ${}_np_x = p_x \cdot p_{x+1} \cdot p_{x+2} \cdot \dots p_{x+n-1}$.

8. What is the probability that a man aged 21 will live 40 years? If he is alive after 20 years, what is the probability that he will then live 20 years more? Compare the two answers.

9. If one house out of 800, of a certain class, is totally destroyed by fire each year, what is the value of the expectation of a man who insures such a house for \$10,000? What should be the net annual cost of such an insurance policy?

10. Using the data of Table X, plot the curve showing the probability of dying for each age. When is it a minimum?

CHAPTER VI

LIFE ANNUITIES

49. Definition of life annuity.—A life annuity in its simplest form, is one whose payments continue only during the lifetime of the individual concerned. Its present value, or cost, will depend not only upon the rate of interest, but also upon the probability of living. In this latter respect it differs from annuities certain, considered in Chapter II; in that case the term, or number of payments, was assumed to be definite.

The present value of a life annuity will be obtained, for any assumed rate of interest, in terms of the probability of survival to any given age. This is accomplished by computing the present value of the expectation at each succeeding age and adding the results.

50. Pure endowment.—The present value of the expectation of a person aged x, who is to receive 1 if he lives to the age $x+n$ is called an *n-year pure endowment of* 1. The present value of 1 due in n years is v^n, and the probability of a person aged x living n years is ${}_np_x$. Denoting the present value of a pure endowment of 1 by ${}_nE_x$, then by the definition

$$ {}_nE_x = v^n \cdot {}_np_x = v^n \cdot \frac{l_{x+n}}{l_x}, \quad . \quad . \quad . \quad . \quad . \quad (1) $$

which is the present value of the expectation, as defined in §46.

Example.—Find the value on a 5 per cent basis, of a pure endowment of \$1000, payable 10 years hence, to a person now aged 25.

The present value is given by

$$ 1000\,{}_{10}E_{25} = 1000v^{10} \cdot {}_{10}p_{25}, $$

where,

$$ {}_{10}p_{25} = \frac{l_{35}}{l_{25}} = \frac{81{,}822}{89{,}032}. $$

Upon performing the indicated operations, it is found that

$$\$1000 \cdot {}_{10}E_{25} = \$564.20.$$

PROBLEMS

1. An estate valued at $25,000 is left to an heir aged 15, to be given him upon his attaining the age of 21. Find the present value of the inheritance upon a 6 per cent basis.

2. Find the present value of a pure endowment of $1000, to a person aged 25, computed upon a 5 per cent basis, (i) payable if he attains the age of 50, (ii) payable if he survives age 75.

3. An heir, aged 12, is to receive $50,000 when he attains the age of 21. What is the present value of his expectation upon a 5 per cent basis?

4. Two brothers, one 15, the other 18, are each to receive $10,000 upon reaching the age of 21. Find the present value of the expectation of each on a 5 per cent basis.

5. Find the values of the expectations in Problem 4, one year later.

51. Computation of life annuity.—From the definition, the present value of a life annuity of 1, to a person aged x, consists of the sum of pure endowments of 1, for each succeeding age. Denoting by a_x the present value, or cost of such an annuity, we have

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \ldots \text{ to table limit}, \quad . \quad . \quad . \quad (2)$$

$$= \frac{vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \ldots \text{ to table limit}}{l_x} \quad . \quad . \quad . \quad (3)$$

This result may also be obtained, without introducing the notion of probability, by supposing that each of a group of l_x people, all aged x, purchase such an annuity. The total amount that must be contributed by the group must be such that each of the survivors, at each succeeding age, may receive 1. Thus, at age $x+1$ there will be l_{x+1} persons alive; hence, an amount vl_{x+1} must be set aside now to provide a sum l_{x+1} at the end of the first year. Similarly, an amount l_{x+2} will have to be paid at the end of the second year, and its present value is $v^2 l_{x+2}$. Proceeding in this manner for each succeeding year, until all of the original l_x persons are dead, we have, as the total amount that must be set aside now to meet the future payments, the sum

$$vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\ .\ .\ .\ \text{to table limit.}$$

This amount, however, is to be shared equally by all of the original l_x persons. Hence, each must contribute

$$\frac{vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\ .\ .\ .}{l_x}$$

which is identical with a_x, as given by Formula (3).

The value of an endowment of R instead of 1, is clearly $R\,{}_nE_x$, and the present value of a life annuity of R is Ra_x.

The labor of computing a_x in the form given by (3) would be very great. This will be lessened, however, by the transformations shown in the next article; but it should be noted that the value of a life annuity at age x can be obtained from the value at age $x+1$ by a simple calculation. The present value of a_{x+1} is va_{x+1}. Hence, the total present cost of l_{x+1} life annuities for the l_{x+1} survivors at age $x+1$, together with the payment of 1 to each at the end of the first year, is

$$vl_{x+1}+va_{x+1}\cdot l_{x+1}.$$

Hence, each of the l_x original persons will have to pay an amount.

$$a_x=v(1+a_{x+1})\frac{l_{x+1}}{l_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus, if the a_{x+1} be known, the value of a_x can be easily found, and indeed, by starting with the most advanced age in the mortality table, a complete table of life annuities could be constructed, for each rate of interest desired.

PROBLEMS

1. From the American Experience Table of Mortality, using $3\frac{1}{2}$ per cent interest, the value of a life annuity of 1 at age 35 is 17.614. Find the value at age 34. How many years, approximately, would an annuity certain have to run to cost the same amount?

2. Given, $a_{50}=13.535$; find, by successive steps, a_{49}, a_{48} and a_{47}. Interest at $3\frac{1}{2}$ per cent.

3. Given, $a_{20}=20.144$; find the value of a_{21}. Interest is at $3\frac{1}{2}$ per cent.

4. By using Formula (3) and the mortality table, find a_{90} on a $3\frac{1}{2}$ per cent basis. Compute it also on a 5 per cent basis.

52. Commutation columns.—While life annuities may be calculated by the preceding methods, the work is greatly facilitated by certain tables known as *commutation columns.*

Starting with a_x, as given by formula (3), we have

$$a_x=\frac{vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\ \ldots\ \text{to table limit}}{l_x}.$$

Multiplying numerator and denominator by v^x, it becomes

$$a_x=\frac{v^{x+1}l_{x+1}+v^{x+2}l_{x+2}+v^{x+3}l_{x+3}+\ \ldots\ \text{to table limit}}{v^xl_x}. \quad (4)$$

If, now, we denote the product $v^{x+k}l_{x+k}$ by the symbol D_{x+k}, we have

$$a_x=\frac{D_{x+1}+D_{x+2}+D_{x+3}+\ \ldots\ \text{to table limit}}{D_x}. \quad . \quad . \quad (5)$$

Placing the numerator equal to N_{x+1} (5) becomes

$$a_x=\frac{N_{x+1}}{D_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$N_{x+1}=D_{x+1}+D_{x+2}+D_{x+3}+\ \ldots\ \text{to table limit}. \quad . \quad (7)$$

The quantities N_x and D_x are known as commutation symbols, and their values for each age are given in Table XI.

53. Deferred annuities.—If the payments of a life annuity are to begin at the end of $n+1$ years, instead of at the end of one year, the annuity is said to be deferred n years.

The cost, or present value, of such an annuity is then less than that of a life annuity whose payments begin the first year by the cost of the endowments for the first n years. Denoting the present value of the life annuity of 1 deferred n years, for a person aged x, by ${}_{n|}a_x$, we have

$$ {}_{n|}a_x = {}_{n+1}E_x + {}_{n+2}E_x + {}_{n+3}E_x + \ldots \text{ to table limit.} \quad . \quad (8)$$

Replacing each one of the E's by its value, given by (1), (8) becomes

$$ {}_{n|}a_x = \frac{v^{n+1}l_{x+n+1} + v^{n+2}l_{x+n+2} + \ldots \text{ to table limit}}{l_x}. \quad . \quad (9)$$

As in § 52, this may be expressed in terms of D's, by multiplying numerator and denominator by v^x. Whence,

$$ {}_{n|}a_x = \frac{D_{x+n+1} + D_{x+n+2} + \ldots \text{ to table limit}}{D_x} \quad . \quad (10)$$

$$ = \frac{N_{x+n+1}}{D_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

54. Temporary annuities.—When the payments of an annuity are to cease at the end of n years, provided the annuitant lives that long, it is called a temporary annuity for n years. It provides for payments during the life of the insured up to the end of n years, but no longer.

The present value of a temporary annuity of 1, for a person of age x, to run n years, is denoted by $a_{x\overline{n|}}$. It is clear that the cost of a temporary annuity for n years, and of one deferred n years, together make up a whole life annuity. Hence,

$$ a_{x\overline{n|}} = a_x - {}_{n|}a_x.$$

$$ = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Table XI gives the values of N_x and of D_x on the basis of the American Experience Table of Mortality, interest being allowed at $3\frac{1}{2}$ per cent. Unless otherwise stated, all computations are assumed to be made on this basis.

PROBLEMS

1. Using Formula (6) and Table XI, find the value of a_{20}.

2. Find the present value of a life annuity to a person aged 50, the annual payment to be \$1000. *Ans.* \$13,534.72.

3. Find the cost of the annuity in Problem 2, taken out at the age of 50, the payments being deferred 10 years. *Ans.* \$5901.04.

4. Find the cost of a temporary annuity of \$1000 per year, taken out at the age of 50, and terminating in 10 years. *Ans.* \$7633.68.

5. What relation exists between the results of Problems 2, 3 and 4? Explain the reason.

6. Find the present value of a life annuity of \$1000 per year, bought by a man aged 40, payments to begin 20 years later.

7. Compute ${}_{20|}a_{30}$ and $a_{30\,\overline{20|}}$.

55. Annuities due.—If the first payment is made at once instead of at the end of the year, and each succeeding payment is also made in advance, the annuity is called an *annuity due.* The present value of a life annuity due of 1, payable annually to a person aged x, is denoted by $\mathbf{a}_x$ (cf. § 19).

An annuity due differs from an ordinary annuity only by the additional first payment. Hence,

$$\mathbf{a}_x = 1 + a_x. \qquad (13)$$

But, from equation (6),

$$a_x = \frac{N_{x+1}}{D_x};$$

hence,

$$\mathbf{a}_x = 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + N_{x+1}}{D_x} = \frac{N_x}{D_x}. \qquad (14)$$

Similarly, the present value of a *deferred* annuity due, the first payment to be made at the beginning of the nth year, is the same as the present value of an ordinary annuity deferred $n-1$ years. Hence, using the corresponding notation,

$${}_{n|}\mathbf{a}_x = {}_{n-1|}a_x = \frac{N_{x+n}}{D_x}. \qquad (15)$$

A *temporary* annuity due, to run n years, is the difference between a whole life annuity due and an n-year deferred life annuity due. Hence,

$$\mathrm{a}_{x\overline{n}|} = \mathrm{a}_x - {}_{n|}\mathrm{a}_x = \frac{N_x - N_{x+n}}{D_x}. \quad . \quad . \quad . \quad . \quad (16)$$

MISCELLANEOUS PROBLEMS

1. Compute a_{30}, a_{30}, ${}_{10|}\mathrm{a}_{30}$ and ${}_{10|}a_{30}$.

2. A will provides that an heir, aged 35, is to receive \$1000 per year, payable in advance, for life, what is the present value of the expected payments?

3. If an estate of \$25,000 is to be turned into cash, and paid in the form of a life annuity to an heir, aged 50, what would be the annual payment?

4. What would the annual payment be in Problem 3, if the annuity were payable for 20 years, contingent upon the survival of the beneficiary?

5. Under the first pension plan of the Carnegie Foundation, the retirement age was 65. This was later changed to 70 for the corresponding pension. Compare the present values of the expectations of a professor aged 35 under the two plans, assuming that his retiring allowance in either case would be \$3000, and payable at the end of each year.

6. A person aged 50 purchases a life annuity for \$25,000. What is the annual income?

7. If, in Problem 6, the payments are to be deferred 10 years, what will be the annual income?

8. How much would a 10-year temporary life annuity yield if purchased at age 55 for \$10,000?

9. A person aged 60, deeded to a university, property valued at \$20,000, in consideration of an equivalent life annuity on a $3\frac{1}{2}$ per cent basis. Find the annual income.

10. A person aged 21 receives an inheritance of \$3000 per year for life, payable at the beginning of each year. An inheritance tax of 4 per cent is to be paid on its present value Find the amount of the tax.

CHAPTER VII

ELEMENTARY PRINCIPLES OF LIFE INSURANCE

56. Introduction.—The mathematical treatment of life insurance involves many complex problems. It will be the purpose of this chapter, however, to consider only certain basic principles upon which the theory rests, and to apply these principles to the determination of the cost of some typical forms of insurance.

The fundamental principle of all insurance is that a large group of persons contribute, through the agency of the insurance companies, for the losses sustained by members of the group. The money paid by the insured for his protection is called the *premium*, and the contract between him and the company is called the *policy*. It is necessary to determine beforehand the amount of the premium sufficient to provide for the probable losses. Indeed, premiums are always collected in advance, usually in annual, or semiannual installments, sometimes even more frequently. It becomes important, therefore, to determine the proper amount to be paid by the policyholder for the particular form of policy desired.

The principal forms of life insurance fall into two classes—*whole-life* and *term* insurance. In whole-life insurance, the company contracts to pay a certain sum upon the policyholder's death. With term insurance, payment is made only if the insured dies within a specified number of years.

The two most important elements that enter into the determination of the premium are the death rate, and the rate of interest that can be realized on investments. When only these two are taken into account, the cost thus determined is called the *net premium*. After this has been determined the company must increase it sufficiently to cover expenses of all kinds. This last process is called *loading*, and the final amount charged for the insurance is called the *gross*, or *office*, premium. The present discussion is concerned only with the determination of the net premium.

The mathemcatical determination of the net premium will be based on the assumptions that deaths will occur with exactly the frequency indicated in the mortality tables, and that earnings will be exactly those resulting from the assumed rate of interest. Furthermore, benefits resulting from deaths in any particular year will be regarded as paid at the *end* of the year. In this connection, *year* means the *policy* year. Twelve calendar months from the date of issue of the policy is the first policy year, the next twelve months is the *second* policy year, and so on.

In all problems connected with life insurance, the results will be derived, as in the case of life annuities, on the basis of a benefit of 1. The solution of the questions that arise are similar to those connected with life annuities, but the probability of dying, rather than that of living, is now under consideration.

57. Net single premium. Whole-life policy.—The present value, or net cost, of a whole-life insurance policy, expressed as a single sum, is known as the *net single premium.*

Using the mortality tables, and assuming a fixed rate of interest, suppose that l_x people, all of age x, pay to an insurance company a sum sufficient to pay 1 for each death as it occurs. The net single premium that each of the l_x persons will have to pay will be the total present value of all future death claims, divided by l_x.

At the end of the first year, the company will be called on to pay an amount d_x; therefore they must now have on hand a sum vd_x, which, with interest, will take care of these claims at the end of the year. Similarly an amount v^2d_{x+1} will have to be provided now to take care of death claims at the end of the second year, and so on until all of the original l_x persons have died. The sum of the present values of all these payments is,

$$vd_x+v^2d_{x+1}+v^3d_{x+2}+\ldots \text{ to table limit.} \quad . \quad . \quad (1)$$

But this cost is to be shared equally by all of the original l_x persons buying insurance. Hence, each would pay an amount which will be denoted by A_x, given by the formula,

$$A_x=\frac{vd_x+v^2d_{x+1}+v^3d_{x+2}+\ldots \text{ to table limit}}{l_x}. \quad . \quad (2)$$

58. Commutation symbols.—Formula (2) could be used for the determination of A_x, but the calculations may be facilitated by the following transformation, analogous to that used in § 52.

Multiplying numerator and denominator by v^x, the denominator becomes D_x, and the terms of the numerator are of the form

$$v^{x+k+1}\cdot d_{x+k}.$$

Defining C_x by the equation,

$$C_x=v^{x+1}d_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we have

$$A_x=\frac{C_x+C_{x+1}+C_{x+2}+\ldots \text{ to table limit}}{D_x}. \quad . \quad . \quad (4)$$

Defining M_x by the equation,

$$M_x=C_x+C_{x+1}+C_{x+2}+\ldots \text{ to table limit,} \quad . \quad . \quad (5)$$

we finally obtain

$$A_x=\frac{M_x}{D_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The values of D_x and M_x, interest at $3\frac{1}{2}$ per cent, are found in Table XI.

PROBLEMS

1. Find the net single premium on a whole-life policy of $1000, on the life of a person aged 30.

2. Find the net single premium for a policy of $1000 on the life of a person aged 70.

3. Compare the costs of whole-life policies of $10,000 for ages 20 and 21, and for ages 70 and 71, noting at which period of life the annual change in cost is the greater.

59. Annual premiums.—It is customary to pay insurance premiums in equal annual installments. They may also be paid semiannually, or even more frequently. Again, they may continue throughout the life of the insured, or they may run for a limited period, say n years, and then cease, even though the insurance continues in the form of a whole-life policy. If the payments continue throughout the life of the insured, the policy is called an *ordinary life policy*. If the premiums are to cease after n years, the policy is called an *n-payment life policy*.

In either of the foregoing cases, the net annual premium is that sum which, if paid at the beginning of each policy year, is equivalent to the net single premium. The annual premiums, therefore, constitute an annuity due, whose present value is A_x.

If P_x denotes the net annual premium for an ordinary life policy, purchased at age x, for an insurance of 1, then, by § 55, formula (13),

$$A_x = P_x\mathbf{a}_x = P_x(1+a_x) = P_x\frac{N_x}{D_x}. \qquad (7)$$

But by (6)

$$A_x = \frac{M_x}{D_x},$$

hence,

$$P_x = \frac{M_x}{N_x}. \qquad (8)$$

The net annual premiums for an n-payment life policy constitute a temporary annuity for $n-1$ years, together with the initial payment at the beginning of the first policy year. Denoting by ${}_nP_x$ the amount of the premium for an insurance of 1, we have

$$ {}_nP_x + {}_nP_x \cdot a_{x\overline{n-1|}} = A_x, $$

or,

$$ {}_nP_x = \frac{A_x}{1+a_{x\overline{n-1|}}}. \qquad (9) $$

By (12), § 54, and (7) § 52,

$$ 1+a_{x\overline{n-1|}} = \frac{N_x - N_{x+n}}{D_x}. $$

Hence, substituting in (9), and replacing A_x by its value, as before, we have

$$ {}_nP_x = \frac{M_x}{N_x - N_{x+n}}. \qquad (10) $$

PROBLEMS

1. Find the net annual premium for an ordinary life policy of $1000, on a life aged 21.

2. Find the net annual premium for an ordinary life policy of $1000, on a life aged 50.

3. Find the net annual premiums for a 20-payment life policy for $1000, for ages 21 and 50 respectively. Compare your answers with those in Problems 1 and 2.

60. Net single premium for term insurance.—As explained in § 56, a term insurance policy is a contract to pay the face of the policy if, and only if, death occurs within the stated term of years. This form of insurance is written for periods of various lengths, but usually for five years, or a multiple of five years. It is usually bought by a person desiring protection for his estate, covering some period within which he is to be engaged in a business enterprise that would suffer loss if he should die before it was sufficiently developed.

The *net single premium* for term insurance of 1 for n years, on the life of a person aged x, will be denoted by the symbol $|_nA_x$. Its value will be found in the same manner as used in § 57 for the determination of A_x.

Suppose that l_x persons, all of age x, purchase n-year term policies. The present value of the death claims for each of the n years will be respectively vd_x, v^2d_{x+1}, ... $v^n d_{x+n-1}$. The sum of these quantities gives the total cost, to be shared equally by the l_x persons buying the insurance. The amount, $|_n A_x$, that each will pay is therefore

$$|_n A_x = \frac{vd_x + v^2 d_{x+1} + \ldots + v^n d_{n+x+1}}{l_x}. \quad . \quad . \qquad (11)$$

If numerator and denominator be multiplied by v^x, the symbols C_x and D_x may be introduced, giving

$$|_n A_x = \frac{C_x + C_{x+1} + C_{x+2} + \ldots + C_{x+n-1}}{D_x}. \quad . \quad . \qquad (12)$$

By (5),

$$M_x = C_x + C_{x+1} + \ldots \text{ to table limit,}$$

and,

$$M_{x+n} = C_{x+n} + C_{x+n+1} + \ldots \text{ to table limit.}$$

Hence, the numerator of (12) is the difference between these two expressions, so that

$$|_n A_x = \frac{M_x - M_{x+n}}{D_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

61. Net annual premium for term insurance.—The net annual premium, $|_n P_x$, may be considered as the annual payments of a temporary annuity due. Hence,

$$|_n P_x \cdot \mathbf{a}_{x\overline{n}|} = |_n A_x. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Substituting the value of $|_n A_x$ from (13), and the value of $\mathbf{a}_{x\overline{n}|}$ from (16) § 55, we have,

$$|_n P_x = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

PROBLEMS

1. Find the net single premium for term insurance of $25,000 for 5 years, on the life of a person aged 40.

2. What would the net annual premium be for the policy in Problem 1?

3. Find the net single premium for term insurance of \$10,000 for 10 years on the life of a person aged 50. Compare it with the net single premium for a whole life policy for the same amount.

4. A person aged 60 buys 5-year term insurance of \$100,000. Find the net annual premium.

62. Endowment insurance.—An endowment insurance is an agreement to pay the face of the policy in the event of the death of the insured within a certain specified period, called the endowment period, and it also provides that the face of the policy will be paid at the end of the period, if the insured survives.

If the period be denoted by n, then endowment insurance consists of term insurance for n years, together with an n-year pure endowment.

Denoting the net single premium for endowment insurance of 1 issued to a person aged x, by $A_{x\overline{n}|}$, we have the relation,

$$A_{x\overline{n}|} = |_n A_x + {}_nE_x. \qquad (16)$$

But, by (1) § 50,

$${}_nE_x = \frac{v^n l_{x+n}}{l_x} = \frac{v^{x+n} l_{x+n}}{v^x l_x}.$$

By definition (§ 52),

$$D_x = v^x l_x,$$

hence,

$${}_nE_x = \frac{D_{x+n}}{D_x}. \qquad (17)$$

Introducing the value of $|_n A_x$ from (13),

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \qquad (18)$$

The *net annual premium* for an n-year endowment policy of 1, for a person aged x, may be obtained by regarding $A_{x\overline{n}|}$ as the present value of an annuity due, to run n years. Hence, if its value be denoted by $P_{x\overline{n}|}$,

$$P_{x\overline{n}|} \cdot \mathrm{a}_{x\overline{n}|} = A_{x\overline{n}|}; \qquad (19)$$

hence,

$$P_{x\overline{n}|}=\frac{M_x-M_{x+n}+D_{x+n}}{N_x-N_{x+n}}. \quad . \quad . \quad . \quad . \quad (20)$$

PROBLEMS

1. Find the net annual premium on a 20-year endowment policy for $10,000, purchased at age 21. What would the premium be if term insurance had been purchased for the same period?

2. Find the net annual premiums on $1000 policies, purchased by a man aged 45, for the following types:

   (*a*) 20-year endowment.
   (*b*) 20-year term.
   (*c*) Whole life.

3. Find the net single premium on a $10,000 endowment policy, purchased at age 25. What would it be if purchased at age 60?

63. Valuation of policies. Reserves.—The probability of dying, except for the very young, increases from year to year. If, then, a person insured himself year by year, his premium would increase with advancing age. It follows, therefore, that when he takes out a whole life policy, paying a uniform or *level* premium throughout, he pays more in the earlier years, and less in the later years, than is required by the natural, or year by year, premium.

The excess paid in the earlier years, over mortality requirements, is held by the company and, with its interest earnings, takes care of the deficiency that will occur in the later years. This amount is known as the *terminal reserve* on the policy, and is a liability of the insurance company to its policyholders.

To determine the terminal reserve at the end of any given year, after the policy has been issued, one notes that the present value, at that time, of the unpaid premiums, together with the terminal reserve, are equal to the net single premium of a policy taken out at the age then attained. If n denotes the number of years since the policy was issued, and ${}_nV_x$ the reserve at that time on a policy issued on a life aged x years, then,

$$A_{x+n} = {}_nV_x + P_x(1 + a_{x+n}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From this equation,

$${}_nV_x = A_{x+n} - P_x(1 + a_{x+n}). \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This formula may be expressed in terms of commutation symbols. From (6),

$$A_{x+n} = \frac{M_{x+n}}{D_{x+n}}.$$

Formula (8) gives

$$P_x = \frac{M_n}{N_x},$$

while (6) and (7) § 52 give

$$1 + a_{x+n} = \frac{N_{x+n}}{D_{x+n}}.$$

Substituting these values in (22),

$${}_nV_x = \frac{M_{x+n} \cdot N_x - M_x \cdot N_{x+n}}{N_x \cdot D_{x+n}}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

PROBLEM

Find the terminal reserve in the fifth policy year on an ordinary life policy of \$1000, taken out at age 21.

64. Gross premium. Loading.—In the preceding articles it has been seen that the net premiums are the mathematical equivalent of the benefits, based, however, on the assumption of a low rate of interest, usually $3\frac{1}{2}$ per cent. Earnings of the insurance company, over and above this rate, go into the general surplus fund, but, in addition to this, it is necessary to provide funds for expenses incident to the business. These include agent's commissions, cost of medical examinations, the general expenses of administration, etc.

These costs are met by increasing, or *loading*, the premium.

This is sometimes done by adding a fixed percentage of the net premium, uniform for all ages; in other cases the percentage varies for different ages. These plans may be also combined with a loading by addition of a constant charge.

The net premium, thus increased by the "loading" process, is called the gross, or office, premium. This is the amount actually paid by the policyholder. Although the methods of loading may differ from company to company, they usually result in substantial agreement for policies of the same kind.

The gross premium thus determined is based on conservative estimates of probable expenditures and claims. In calculating the net premium, a low rate of interest is assumed. The earnings of the company will, in general, be substantially larger than those estimated at this rate. The mortality table is also constructed so as to give conservative results with respect to probable death claims, and the loading for expenses incident to the business is designed to cover adequately all such expenditures.

At the end of each year, then, there should be a *surplus* remaining, after all expenses have been met and funds have been set aside to meet all reserves, which, as already seen, constitute a liability against the company. In the mutual companies, this surplus belongs to the policyholders, and is returned to them in the form of *dividends*, or credited to the policy as additional insurance.

65. Conclusion.—The purpose of the last two chapters has been to show what mathematical principles enter into the fundamental problems of life insurance. These have been formulated in as simple a manner as possible, but the matters presented are merely an introduction to the subject.

Students interested in a further study of insurance are referred to such books as Moir's "Life Assurance Primer," and the "Institute of Actuaries Text-Book."

TABLE I—THE NUMBER OF EACH DAY OF THE YEAR

Day of Month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29		88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

Note.—For leap years the number of the day is one greater than the tabular number after February 28.

TABLE II—AMOUNT OF 1

$s=(1+i)^n$

n	½%	1%	1¼%	1½%	n
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1
2	1.010 0250	1.020 1000	1.025 1562	1.030 2250	2
3	1.015 0751	1.030 3010	1.037 9707	1.045 6784	3
4	1.020 1505	1.040 6040	1.050 9453	1.061 3636	4
5	1.025 2512	1.051 0100	1.064 0822	1.077 2840	5
6	1.030 3775	1.061 5202	1.077 3832	1.093 4433	6
7	1.035 5294	1.072 1354	1.090 8505	1.109 8449	7
8	1.040 7070	1.082 8567	1.104 4861	1.126 4926	8
9	1.045 9106	1.093 6853	1.118 2922	1.143 3900	9
10	1.051 1401	1.104 6221	1.132 2708	1.160 5408	10
11	1.056 3958	1.115 6684	1.146 4242	1.177 9489	11
12	1.061 6778	1.126 8250	1.160 7545	1.195 6182	12
13	1.066 9862	1.138 0933	1.175 2640	1.213 5524	13
14	1.072 3211	1.149 4742	1.189 9548	1.231 7557	14
15	1.077 6827	1.160 9690	1.204 8292	1.250 2321	15
16	1.083 0712	1.172 5786	1.219 8896	1.268 9856	16
17	1.088 4865	1.184 3044	1.235 1382	1.288 0203	17
18	1.093 9289	1.196 1475	1.250 5774	1.307 3406	18
19	1.099 3986	1.208 1090	1.266 2096	1.326 9508	19
20	1.104 8956	1.220 1900	1.282 0372	1.346 8550	20
21	1.110 4201	1.232 3919	1.298 0627	1.367 0578	21
22	1.115 9722	1.244 7159	1.314 2885	1.387 5637	22
23	1.121 5520	1.257 1630	1.330 7171	1.408 3772	23
24	1.127 1598	1.269 7346	1.347 3510	1.429 5028	24
25	1.132 7956	1.282 4320	1.364 1929	1.450 9454	25
26	1.138 4596	1.295 2563	1.381 2454	1.472 7095	26
27	1.144 1518	1.308 2089	1.398 5109	1.494 8002	27
28	1.149 8726	1.321 2910	1.415 9923	1.517 2222	28
29	1.155 6220	1.334 5039	1.433 6922	1.539 9805	29
30	1.161 4001	1.347 8489	1.451 6134	1.563 0802	30
31	1.167 2071	1.361 3274	1.469 7585	1.586 5264	31
32	1.173 0431	1.374 9407	1.488 1305	1.610 3243	32
33	1.178 9083	1.388 6901	1.506 7321	1.634 4792	33
34	1.184 8029	1.402 5770	1.525 5663	1.658 9964	34
35	1.190 7269	1.416 6028	1.544 6359	1.683 8813	35
36	1.196 6805	1.430 7688	1.563 9438	1.709 1395	36
37	1.202 6639	1.445 0765	1.583 4931	1.734 7766	37
38	1.208 6772	1.459 5272	1.603 2868	1.760 7983	38
39	1.214 7206	1.474 1225	1.623 3279	1.787 2102	39
40	1.220 7942	1.488 8637	1.643 6195	1.814 0184	40
41	1.226 8982	1.503 7524	1.664 1647	1.841 2287	41
42	1.233 0327	1.518 7899	1.684 9668	1.868 8471	42
43	1.239 1979	1.533 9778	1.706 0288	1.896 8798	43
44	1.245 3938	1.549 3176	1.727 3542	1.925 3330	44
45	1.251 6208	1.564 8108	1.748 9461	1.954 2130	45
46	1.257 8789	1.580 4588	1.770 8080	1.983 5262	46
47	1.264 1683	1.596 2634	1.792 9431	2.013 2791	47
48	1.270 4892	1.612 2261	1.815 3548	2.043 4783	48
49	1.276 8416	1.628 3483	1.838 0468	2.074 1305	49
50	1.283 2258	1.644 6318	1.861 0224	2.105 2424	50
60	1.348 8502	1.816 6967	2.107 1814	2.443 2198	60
70	1.417 8305	2.006 7634	2.385 9000	2.835 4563	70
80	1.490 3386	2.216 7152	2.701 4849	3.290 6628	80
90	1.566 5547	2.448 6327	3.058 8126	3.818 9485	90
100	1.646 6685	2.704 8138	3.463 4043	4.432 0456	100

TABLE II—AMOUNT OF 1—*Continued*

$$s = (1 + i)^n$$

n	1¾%	2%	2½%	3%	n
1	1.017 5000	1.020 0000	1.025 0000	1.030 0000	1
2	1.035 3062	1.040 4000	1.050 6250	1.060 9000	2
3	1.053 4241	1.061 2080	1.076 8906	1.092 7270	3
4	1.071 8590	1.082 4322	1.103 8129	1.125 5088	4
5	1.090 6166	1.104 0808	1.131 4082	1.159 2741	5
6	1.109 7024	1.126 1624	1.159 6934	1.194 0523	6
7	1.129 1222	1.148 6857	1.188 6858	1.229 8739	7
8	1.148 8818	1.171 6594	1.218 4029	1.266 7701	8
9	1.168 9872	1.195 0926	1.248 8630	1.304 7732	9
10	1.189 4445	1.218 9944	1.280 0845	1.343 9164	10
11	1.210 2598	1.243 3743	1.312 0867	1.384 2339	11
12	1.231 4393	1.268 2418	1.344 8888	1.425 7609	12
13	1.252 9895	1.293 6066	1.378 5110	1.468 5337	13
14	1.274 9168	1.319 4788	1.412 9738	1.512 5897	14
15	1.297 2279	1.345 8683	1.448 2982	1.557 9674	15
16	1.319 9294	1.372 7857	1.484 5056	1.604 7064	16
17	1.343 0281	1.400 2414	1.521 6183	1.652 8476	17
18	1.366 5311	1.428 2462	1.559 6587	1.702 4331	18
19	1.390 4454	1.456 8112	1.598 6502	1.753 5060	19
20	1.414 7782	1.485 9474	1.638 6164	1.806 1112	20
21	1.439 5368	1.515 6663	1.679 5818	1.860 2946	21
22	1.464 7287	1.545 9797	1.721 5714	1.916 1034	22
23	1.490 3615	1.576 8993	1.764 6107	1.973 5865	23
24	1.516 4428	1.608 4372	1.808 7260	2.032 7941	24
25	1.542 9805	1.640 6060	1.853 9441	2.093 7779	25
26	1.569 9827	1.673 4181	1.900 2927	2.156 5913	26
27	1.597 4574	1.706 8865	1.947 8000	2.221 2890	27
28	1.625 4129	1.741 0242	1.996 4950	2.287 9277	28
29	1.653 8576	1.775 8447	2.046 4074	2.356 5655	29
30	1.682 8001	1.811 3616	2.097 5676	2.427 2625	30
31	1.712 2491	1.847 5888	2.150 0068	2.500 0804	31
32	1.742 2135	1.884 5406	2.203 7569	2.575 0828	32
33	1.772 7022	1.922 2314	2.258 8509	2.652 3352	33
34	1.803 7245	1.960 6760	2.315 3221	2.731 9053	34
35	1.835 2897	1.999 8896	2.373 2052	2.813 8624	35
36	1.867 4073	2.039 8873	2.432 5353	2.898 2783	36
37	1.900 0869	2.080 6851	2.493 3487	2.985 2267	37
38	1.933 3384	2.122 2988	2.555 6824	3.074 7835	38
39	1.967 1718	2.164 7448	2.619 5745	3.167 0270	39
40	2.001 5973	2.208 0397	2.685 0638	3.262 0378	40
41	2.036 6253	2.252 2005	2.752 1904	3.359 8989	41
42	2.072 2662	2.297 2445	2.820 9952	3.460 6959	42
43	2.108 5309	2.343 1894	2.891 5201	3.564 5168	43
44	2.145 4302	2.390 0531	2.963 8081	3.671 4523	44
45	2.182 9752	2.437 8542	3.037 9033	3.781 5958	45
46	2.221 1773	2.486 6113	3.113 8509	3.895 0437	46
47	2.260 0479	2.536 3435	3.191 6971	4.011 8950	47
48	2.299 5987	2.587 0704	3.271 4896	4.132 2519	48
49	2.339 8417	2.638 8118	3.353 2768	4.256 2194	49
50	2.380 7889	2.691 5880	3.437 1087	4.383 9060	50
60	2.831 8163	3.281 0308	4.399 7898	5.891 6031	60
70	3.368 2883	3.999 5582	5.632 1029	7.917 8219	70
80	4.006 3919	4.875 4392	7.209 5678	10.640 8906	80
90	4.765 3808	5.943 1331	9.228 8563	14.300 4671	90
100	5.668 1559	7.244 6461	11.813 7164	19.218 6320	100

TABLE II—AMOUNT OF 1—*Continued*

$s = (1 + i)^n$

n	3½%	4%	4½%	4¾%	n
1	1.035 0000	1.040 0000	1.045 0000	1.047 5000	1
2	1.071 2250	1.081 6000	1.092 0250	1.097 2562	2
3	1.108 7179	1.124 8640	1.141 1661	1.149 3759	3
4	1.147 5230	1.169 8586	1.192 5186	1.203 9713	4
5	1.187 6863	1.216 6529	1.246 1819	1.261 1599	5
6	1.229 2553	1.265 3190	1.302 2601	1.321 0650	6
7	1.272 2793	1.315 9318	1.360 8618	1.383 8156	7
8	1.316 8090	1.368 5690	1.422 1006	1.449 5468	8
9	1.362 8974	1.423 3118	1.486 0951	1.518 4003	9
10	1.410 5988	1.480 2443	1.552 9694	1.590 5243	10
11	1.459 9697	1.539 4541	1.622 8530	1.666 0742	11
12	1.511 0687	1.601 0322	1.695 8814	1.745 2128	12
13	1.563 9561	1.665 0735	1.772 1961	1.828 1104	13
14	1.618 6945	1.731 6764	1.851 9449	1.914 9456	14
15	1.675 3488	1.800 9435	1.935 2824	2.005 9055	15
16	1.733 9860	1.872 9812	2.022 3702	2.101 1860	16
17	1.794 6756	1.947 9005	2.113 3768	2.200 9924	17
18	1.857 4892	2.025 8165	2.208 4788	2.305 5395	18
19	1.922 5013	2.106 8492	2.307 8603	2.415 0526	19
20	1.989 7889	2.191 1231	2.411 7140	2.529 7676	20
21	2.059 4315	2.278 7681	2.520 2412	2.649 9316	21
22	2.131 5116	2.369 9188	2.633 6520	2.775 8034	22
23	2.206 1145	2.464 7155	2.752 1664	2.907 6540	23
24	2.283 3285	2.563 3042	2.876 0138	3.045 7676	24
25	2.363 2450	2.665 8363	3.005 4345	3.190 4415	25
26	2.445 9586	2.772 4698	3.140 6790	3.341 9875	26
27	2.531 5671	2.883 3686	3.282 0096	3.500 7319	27
28	2.620 1720	2.998 7033	3.429 7000	3.667 0167	28
29	2.711 8780	3.118 6514	3.584 0365	3.841 2000	29
30	2.806 7937	3.243 3975	3.745 3181	4.023 6570	30
31	2.905 0315	3.373 1334	3.913 8574	4.214 7807	31
32	3.006 7076	3.508 0588	4.089 9810	4.414 9828	32
33	3.111 9424	3.648 3811	4.274 0302	4.624 6944	33
34	3.220 8603	3.794 3163	4.466 3615	4.844 3674	34
35	3.333 5904	3.946 0890	4.667 3478	5.074 4749	35
36	3.450 2661	4.103 9326	4.877 3785	5.315 5124	36
37	3.571 0254	4.268 0899	5.096 8605	5.567 9993	37
38	3.696 0113	4.438 8134	5.326 2192	5.832 4792	38
39	3.825 3717	4.616 3660	5.565 8991	6.109 5220	39
40	3.959 2597	4.801 0206	5.816 3645	6.399 7243	40
41	4.097 8338	4.993 0614	6.078 1009	6.703 7112	41
42	4.241 2580	5.192 7839	6.351 6155	7.022 1375	42
43	4.389 7020	5.400 4953	6.637 4382	7.355 6890	43
44	4.543 3416	5.616 5151	6.936 1229	7.705 0843	44
45	4.702 3586	5.841 1757	7.248 2484	8.071 0758	45
46	4.866 9411	6.074 8227	7.574 4196	8.454 4519	46
47	5.037 2840	6.317 8156	7.915 2685	8.856 0383	47
48	5.213 5890	6.570 5282	8.271 4556	9.276 7001	48
49	5.396 0646	6.833 3494	8.643 6711	9.717 3434	49
50	5.584 9269	7.106 6834	9.032 6363	10.178 9172	50
60	7.878 0909	10.519 6274	14.027 4079	16.189 8154	60
70	11.112 8253	15.571 6184	21.784 1356	25.750 2954	70
80	15.675 7375	23.049 7991	33.830 0964	40.956 4712	80
90	22.112 1760	34.119 3333	52.537 1053	65.142 2639	90
100	31.191 4080	50.504 9482	81.588 5180	103.610 3555	100

TABLE II—AMOUNT OF 1—*Continued*

$$s = (1 + i)^n$$

n	5%	6%	7%	8%	n
1	1.050 0000	1.060 0000	1.070 0000	1.080 0000	**1**
2	1.102 5000	1.123 6000	1.144 9000	1.166 4000	**2**
3	1.157 6250	1.191 0160	1.225 0430	1.259 7120	**3**
4	1.215 5062	1.262 4770	1.310 7960	1.360 4890	**4**
5	1.276 2816	1.338 2256	1.402 5517	1.469 3281	**5**
6	1.340 0956	1.418 5191	1.500 7304	1.586 8743	**6**
7	1.407 1004	1.503 6303	1.605 7815	1.713 8243	**7**
8	1.477 4554	1.593 8481	1.718 1862	1.850 9302	**8**
9	1.551 3282	1.689 4790	1.838 4592	1.999 0046	**9**
10	1.628 8946	1.790 8477	1.967 1514	2.158 9250	**10**
11	1.710 3394	1.898 2986	2.104 8520	2.331 6390	**11**
12	1.795 8563	2.012 1965	2.252 1916	2.518 1701	**12**
13	1.885 6491	2.132 9283	2.409 8450	2.719 6237	**13**
14	1.979 9316	2.260 9040	2.578 5342	2.937 1936	**14**
15	2.078 9282	2.396 5582	2.759 0315	3.172 1691	**15**
16	2.182 8746	2.540 3517	2.952 1638	3.425 9426	**16**
17	2.292 0183	2.692 7728	3.158 8152	3.700 0180	**17**
18	2.406 6192	2.854 3392	3.379 9323	3.996 0195	**18**
19	2.526 9502	3.025 5995	3.616 5275	4.315 7011	**19**
20	2.653 2977	3.207 1355	3.869 6845	4.660 9571	**20**
21	2.785 9626	3.399 5636	4.140 5624	5.033 8337	**21**
22	2.925 2607	3.603 5374	4.430 4017	5.436 5404	**22**
23	3.071 5238	3.819 7497	4.740 5299	5.871 4636	**23**
24	3.225 0999	4.048 9346	5.072 3670	6.341 1807	**24**
25	3.386 3549	4.291 8707	5.427 4326	6.848 4752	**25**
26	3.555 6727	4.549 3830	5.807 3529	7.396 3532	**26**
27	3.733 4563	4.822 3459	6.213 8676	7.988 0615	**27**
28	3.920 1291	5.111 6867	6.648 8384	8.627 1064	**28**
29	4.116 1356	5.418 3879	7.114 2570	9.317 2749	**29**
30	4.321 9424	5.743 4912	7.612 2550	10.062 6569	**30**
31	4.538 0395	6.088 1006	8.145 1129	10.867 6694	**31**
32	4.764 9415	6.453 3867	8.715 2708	11.737 0830	**32**
33	5.003 1885	6.840 5899	9.325 3398	12.676 0496	**33**
34	5.253 3480	7.251 0253	9.978 1135	13.690 1336	**34**
35	5.516 0154	7.686 0868	10.676 5815	14.785 3443	**35**
36	5.791 8161	8.147 2520	11.423 9422	15.968 1718	**36**
37	6.081 4069	8.636 0871	12.223 6181	17.245 6256	**37**
38	6.385 4773	9.154 2524	13.079 2714	18.625 2756	**38**
39	6.704 7512	9.703 5075	13.994 8204	20.115 2977	**39**
40	7.039 9887	10.285 7179	14.974 4578	21.724 5215	**40**
41	7.391 9882	10.902 8610	16.022 6699	23.462 4832	**41**
42	7.761 5876	11.557 0327	17.144 2568	25.339 4819	**42**
43	8.149 6669	12.250 4546	18.344 3548	27.366 6404	**43**
44	8.557 1503	12.985 4819	19.628 4596	29.555 9717	**44**
45	8.985 0078	13.764 6108	21.002 4518	31.920 4494	**45**
46	9.434 2582	14.590 4875	22.472 6234	34.474 0853	**46**
47	9.905 9711	15.465 9167	24.045 7070	37.232 0122	**47**
48	10.401 2696	16.393 8717	25.728 9065	40.210 5731	**48**
49	10.921 3331	17.377 5040	27.529 9300	43.427 4190	**49**
50	11.467 3998	18.420 1543	29.547 0251	46.901 6125	**50**
60	18.679 1859	32.987 6908	57.946 4268	101.257 0637	**60**
70	30.426 4255	59.075 9302	113.989 3922	218.606 4059	**70**
80	49.561 4411	105.795 9935	224.234 3876	471.954 8343	**80**
90	80.730 3650	189.464 5112	441.102 9799	1018.915 0893	**90**
100	131.501 2578	339.302 0835	867.716 3256	2199.761 2563	**100**

TABLE III—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	½%	1%	1¼%	1½%	n
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	1
2	0.990 0745	0.980 2960	0.975 4611	0.970 6618	2
3	0.985 1488	0.970 5902	0.963 4183	0.956 3170	3
4	0.980 2475	0.960 9803	0.951 5243	0.942 1842	4
5	0.975 3707	0.951 4657	0.939 7771	0.928 2603	5
6	0.970 5181	0.942 0452	0.928 1749	0.914 5422	6
7	0.965 6896	0.932 7180	0.916 7159	0.901 0268	7
8	0.960 8852	0.923 4832	0.905 3984	0.887 7111	8
9	0.956 1047	0.914 3398	0.894 2207	0.874 5922	9
10	0.951 3479	0.905 2870	0.883 1809	0.861 6672	10
11	0.946 6149	0.896 3237	0.872 2775	0.848 9332	11
12	0.941 9053	0.887 4492	0.861 5086	0.836 3874	12
13	0.937 2192	0.878 6626	0.850 8727	0.824 0270	13
14	0.932 5565	0.869 9630	0.840 3681	0.811 8493	14
15	0.927 9169	0.861 3495	0.829 9932	0.799 8515	15
16	0.923 3004	0.852 8213	0.819 7464	0.788 0310	16
17	0.918 7068	0.844 3775	0.809 6260	0.776 3853	17
18	0.914 1362	0.836 0173	0.799 6306	0.764 9116	18
19	0.909 5882	0.827 7399	0.789 7587	0.753 6075	19
20	0.905 0629	0.819 5445	0.780 0086	0.742 4704	20
21	0.900 5601	0.811 4302	0.770 3788	0.731 4980	21
22	0.896 0797	0.803 3962	0.760 8680	0.720 6876	22
23	0.891 6216	0.795 4418	0.751 4745	0.710 0371	23
24	0.887 1857	0.787 5661	0.742 1971	0.699 5439	24
25	0.882 7718	0.779 7684	0.733 0341	0.689 2058	25
26	0.878 3799	0.772 0480	0.723 9843	0.679 0205	26
27	0.874 0099	0.764 4039	0.715 0463	0.668 9857	27
28	0.869 6616	0.756 8356	0.706 2185	0.659 0992	28
29	0.865 3349	0.749 3422	0.697 4998	0.649 3589	29
30	0.861 0297	0.741 9229	0.688 8887	0.639 7624	30
31	0.856 7460	0.734 5772	0.680 3839	0.630 3078	31
32	0.852 4836	0.727 3041	0.671 9841	0.620 9929	32
33	0.848 2424	0.720 1031	0.663 6880	0.611 8157	33
34	0.844 0223	0.712 9733	0.655 4943	0.602 7741	34
35	0.839 8231	0.705 9142	0.647 4018	0.593 8661	35
36	0.835 6449	0.698 9250	0.639 4092	0.585 0897	36
37	0.831 4875	0.692 0049	0.631 5152	0.576 4431	37
38	0.827 3507	0.685 1534	0.623 7187	0.567 9242	38
39	0.823 2346	0.678 3697	0.616 0185	0.559 5313	39
40	0.819 1389	0.671 6531	0.608 4133	0.551 2623	40
41	0.815 0635	0.665 0031	0.600 9021	0.543 1156	41
42	0.811 0085	0.658 4189	0.593 4835	0.535 0892	42
43	0.806 9736	0.651 8999	0.586 1566	0.527 1815	43
44	0.802 9588	0.645 4455	0.578 9201	0.519 3907	44
45	0.798 9640	0.639 0549	0.571 7729	0.511 7149	45
46	0.794 9891	0.632 7276	0.564 7140	0.504 1526	46
47	0.791 0339	0.626 4630	0.557 7422	0.496 7021	47
48	0.787 0984	0.620 2604	0.550 8565	0.489 3617	48
49	0.783 1825	0.614 1192	0.544 0558	0.482 1298	49
50	0.779 2861	0.608 0388	0.537 3390	0.475 0047	50
60	0.741 3722	0.550 4496	0.474 5676	0.409 2960	60
70	0.705 3029	0.498 3149	0.419 1290	0.352 6769	70
80	0.670 9885	0.451 1179	0.370 1668	0.303 8902	80
90	0.638 3435	0.408 3912	0.326 9242	0.261 8522	90
100	0.607 2868	0.369 7112	0.288 7333	0.225 6294	100

TABLE III—PRESENT VALUE OF 1—*Continued*

$$v^n = (1+i)^{-n}$$

n	1¾%	2%	2½%	3%	n
1	0.982 8010	0.980 3922	0.975 6098	0.970 8738	1
2	0.965 8978	0.961 1688	0.951 8144	0.942 5959	2
3	0.949 2853	0.942 3223	0.928 5994	0.915 1417	3
4	0.932 9585	0.923 8454	0.905 9506	0.888 4870	4
5	0.916 9125	0.905 7308	0.883 8543	0.862 6088	5
6	0.901 1425	0.887 9714	0.862 2969	0.837 4843	6
7	0.885 6438	0.870 5602	0.841 2652	0.813 0915	7
8	0.870 4116	0.853 4904	0.820 7466	0.789 4092	8
9	0.855 4414	0.836 7553	0.800 7284	0.766 4167	9
10	0.840 7286	0.820 3483	0.781 1984	0.744 0939	10
11	0.826 2689	0.804 2630	0.762 1448	0.722 4213	11
12	0.812 0579	0.788 4932	0.743 5559	0.701 3799	12
13	0.798 0913	0.773 0325	0.725 4204	0.680 9513	13
14	0.784 3649	0.757 8750	0.707 7272	0.661 1178	14
15	0.770 8746	0.743 0147	0.690 4656	0.641 8620	15
16	0.757 6163	0.728 4458	0.673 6249	0.623 1669	16
17	0.744 5860	0.714 1626	0.657 1951	0.605 0164	17
18	0.731 7799	0.700 1594	0.641 1659	0.587 3946	18
19	0.719 1940	0.686 4308	0.625 5277	0.570 2860	19
20	0.706 8246	0.672 9713	0.610 2709	0.553 6758	20
21	0.694 6679	0.659 7758	0.595 3863	0.537 5493	21
22	0.682 7203	0.646 8390	0.580 8647	0.521 8925	22
23	0.670 9782	0.634 1559	0.566 6972	0.506 6918	23
24	0.659 4380	0.621 7215	0.552 8754	0.491 9337	24
25	0.648 0963	0.609 5309	0.539 3906	0.477 6056	25
26	0.636 9497	0.597 5793	0.526 2347	0.463 6947	26
27	0.625 9948	0.585 8620	0.513 3997	0.450 1891	27
28	0.615 2283	0.574 3746	0.500 8778	0.437 0768	28
29	0.604 6470	0.563 1123	0.488 6612	0.424 3464	29
30	0.594 2476	0.552 0709	0.476 7427	0.411 9868	30
31	0.584 0272	0.541 2460	0.465 1148	0.399 9872	31
32	0.573 9825	0.530 6333	0.453 7706	0.388 3370	32
33	0.564 1105	0.520 2287	0.442 7030	0.377 0262	33
34	0.554 4084	0.510 0282	0.431 9053	0.366 0449	34
35	0.544 8731	0.500 0276	0.421 3711	0.355 3834	35
36	0.535 5018	0.490 2232	0.411 0937	0.345 0324	36
37	0.526 2917	0.480 6109	0.401 0670	0.334 9829	37
38	0.517 2400	0.471 1872	0.391 2849	0.325 2262	38
39	0.508 3440	0.461 9482	0.381 7414	0.315 7536	39
40	0.499 6010	0.452 8904	0.372 4306	0.306 5568	40
41	0.491 0083	0.444 0102	0.363 3470	0.297 6280	41
42	0.482 5635	0.435 3041	0.354 4848	0.288 9592	42
43	0.474 2639	0.426 7688	0.345 8389	0.280 5429	43
44	0.466 1070	0.418 4007	0.337 4038	0.272 3718	44
45	0.458 0904	0.410 1968	0.329 1744	0.264 4386	45
46	0.450 2117	0.402 1537	0.321 1458	0.256 7365	46
47	0.442 4685	0.394 2684	0.313 3129	0.249 2588	47
48	0.434 8585	0.386 5376	0.305 6712	0.241 9988	48
49	0.427 3793	0.378 9584	0.298 2158	0.234 9503	49
50	0.420 0288	0.371 5279	0.290 9422	0.228 1071	50
60	0.353 1302	0.304 7823	0.227 2836	0.169 7331	60
70	0.296 8867	0.250 0276	0.177 5536	0.126 2974	70
80	0.249 60[illegible]1	0.205 1097	0.138 7046	0.093 9771	80
90	0.209 8468	0.168 2614	0.108 3558	0.069 9278	90
100	0.176 4242	0.138 0330	0.084 6474	0.052 0328	100

TABLE III—PRESENT VALUE OF 1—*Continued*

$$v^n = (1+i)^{-n}$$

n	3½%	4%	4½%	4¾%	n
1	0.966 1836	0.961 5385	0.956 9378	0.954 6539	1
2	0.933 5107	0.924 5562	0.915 7300	0.911 3641	2
3	0.901 9427	0.888 9964	0.876 2966	0.870 0374	3
4	0.871 4422	0.854 8042	0.838 5613	0.830 5846	4
5	0.841 9732	0.821 9271	0.802 4510	0.792 9209	5
6	0.813 5006	0.790 3145	0.767 8957	0.756 9650	6
7	0.785 9910	0.759 9178	0.734 8285	0.722 6396	7
8	0.759 4116	0.730 6902	0.703 1851	0.689 8708	8
9	0.733 7310	0.702 5867	0.672 9044	0.658 5878	9
10	0.708 9188	0.675 5642	0.643 9277	0.628 7235	10
11	0.684 9457	0.649 5809	0.616 1987	0.600 2134	11
12	0.661 7833	0.624 5970	0.589 6639	0.572 9960	12
13	0.639 4042	0.600 5741	0.564 2716	0.547 0129	13
14	0.617 7818	0.577 4751	0.539 9729	0.522 2080	14
15	0.596 8906	0.555 2645	0.516 7204	0.498 5280	15
16	0.576 7059	0.533 9082	0.494 4693	0.475 9217	16
17	0.557 2038	0.513 3732	0.473 1764	0.454 3405	17
18	0.538 3611	0.493 6281	0.452 8004	0.433 7380	18
19	0.520 1557	0.474 6424	0.433 3018	0.414 0696	19
20	0.502 5659	0.456 3870	0.414 6429	0.395 2932	20
21	0.485 5709	0.438 8336	0.396 7874	0.377 3682	21
22	0.469 1506	0.421 9554	0.379 7009	0.360 2561	22
23	0.453 2856	0.405 7263	0.363 3501	0.343 9199	23
24	0.437 9571	0.390 1215	0.347 7035	0.328 3245	24
25	0.423 1470	0.375 1168	0.332 7306	0.313 4362	25
26	0.408 8377	0.360 6892	0.318 4025	0.299 2231	26
27	0.395 0122	0.346 8166	0.304 6914	0.285 6546	27
28	0.381 6543	0.333 4775	0.291 5707	0.272 7012	28
29	0.368 7482	0.320 6514	0.279 0150	0.260 3353	29
30	0.356 2784	0.308 3187	0.267 0000	0.248 5301	30
31	0.344 2304	0.296 4603	0.255 5024	0.237 2603	31
32	0.332 5897	0.285 0579	0.244 4999	0.226 5014	32
33	0.321 3427	0.274 0942	0.233 9712	0.216 2305	33
34	0.310 4760	0.263 5521	0.223 8959	0.206 4253	34
35	0.299 9769	0.253 4155	0.214 2544	0.197 0647	35
36	0.289 8327	0.243 6687	0.205 0282	0.188 1286	36
37	0.280 0316	0.234 2968	0.196 1992	0.179 5977	37
38	0.270 5619	0.225 2854	0.187 7504	0.171 4537	38
39	0.261 4125	0.216 6206	0.179 6655	0.163 6789	39
40	0.252 5725	0.208 2890	0.171 9287	0.156 2567	40
41	0.244 0314	0.200 2779	0.164 5251	0.149 1711	41
42	0.235 7791	0.192 5749	0.157 4403	0.142 4068	42
43	0.227 8059	0.185 1682	0.150 6605	0.135 9492	43
44	0.220 1023	0.178 0464	0.144 1728	0.129 7844	44
45	0.212 6592	0.171 1984	0.137 9644	0.123 8992	45
46	0.205 4679	0.164 6139	0.132 0233	0.118 2809	46
47	0.198 5197	0.158 2826	0.126 3381	0.112 9173	47
48	0.191 8064	0.152 1948	0.120 8977	0.107 7970	48
49	0.185 3202	0.146 3411	0.115 6916	0.102 9088	49
50	0.179 0534	0.140 7126	0.110 7096	0.098 2423	50
60	0.126 9343	0.095 0604	0.071 2890	0.061 7672	60
70	0.089 9861	0.064 2194	0.045 9050	0.038 8345	70
80	0.063 7928	0.043 3843	0.029 5595	0.024 4162	80
90	0.045 2240	0.029 3089	0.019 0342	0.015 3510	90
100	0.032 0601	0.019 8000	0.012 2566	0.009 6515	100

TABLE III—PRESENT VALUE OF 1—*Continued*

$$v^n = (1 + i)^{-n}$$

n	5%	6%	7%	8%	n
1	0.952 3810	0.943 3962	0.934 5794	0.925 9259	1
2	0.907 0295	0.889 9964	0.873 4387	0.857 3388	2
3	0.863 8376	0.839 6193	0.816 2979	0.793 8322	3
4	0.822 7025	0.792 0937	0.762 8952	0.735 0298	4
5	0.783 5262	0.747 2582	0.712 9862	0.680 5832	5
6	0.746 2154	0.704 9605	0.666 3422	0.630 1696	6
7	0.710 6813	0.665 0571	0.622 7497	0.583 4904	7
8	0.676 8394	0.627 4124	0.582 0091	0.540 2689	8
9	0.644 6089	0.591 8985	0.543 9337	0.500 2490	9
10	0.613 9132	0.558 3948	0.508 3493	0.463 1935	10
11	0.584 6793	0.526 7875	0.475 0928	0.428 8829	11
12	0.556 8374	0.496 9694	0.444 0120	0.397 1138	12
13	0.530 3214	0.468 8390	0.414 9644	0.367 6979	13
14	0.505 0680	0.442 3010	0.387 8172	0.340 4610	14
15	0.481 0171	0.417 2651	0.362 4460	0.315 2417	15
16	0.458 1115	0.393 6463	0.338 7346	0.291 8905	16
17	0.436 2967	0.371 3644	0.316 5744	0.270 2690	17
18	0.415 5206	0.350 3438	0.295 8639	0.250 2490	18
19	0.395 7340	0.330 5130	0.276 5083	0.231 7121	19
20	0.376 8895	0.311 8047	0.258 4190	0.214 5482	20
21	0.358 9424	0.294 1554	0.241 5131	0.198 6558	21
22	0.341 8499	0.277 5051	0.225 7132	0.183 9405	22
23	0.325 5713	0.261 7973	0.210 9469	0.170 3153	23
24	0.310 0679	0.246 9786	0.197 1466	0.157 6993	24
25	0.295 3028	0.232 9986	0.184 2492	0.146 0179	25
26	0.281 2407	0.219 8100	0.172 1955	0.135 2018	26
27	0.267 8483	0.207 3680	0.160 9304	0.125 1868	27
28	0.255 0936	0.195 6301	0.150 4022	0.115 9137	28
29	0.242 9463	0.184 5567	0.140 5628	0.107 3275	29
30	0.231 3774	0.174 1101	0.131 3671	0.099 3773	30
31	0.220 3595	0.164 2548	0.122 7730	0.092 0160	31
32	0.209 8662	0.154 9574	0.114 7411	0.085 2000	32
33	0.199 8725	0.146 1862	0.107 2347	0.078 8889	33
34	0.190 3548	0.137 9115	0.100 2193	0.073 0453	34
35	0.181 2903	0.130 1052	0.093 6629	0.067 6345	35
36	0.172 6574	0.122 7408	0.087 5355	0.062 6246	36
37	0.164 4356	0.115 7932	0.081 8088	0.057 9857	37
38	0.156 6054	0.109 2388	0.076 4569	0.053 6905	38
39	0.149 1480	0.103 0555	0.071 4550	0.049 7134	39
40	0.142 0457	0.097 2222	0.066 7804	0.046 0309	40
41	0.135 2816	0.091 7190	0.062 4116	0.042 6212	41
42	0.128 8396	0.086 5274	0.058 3286	0.039 4641	42
43	0.122 7044	0.081 6296	0.054 5127	0.036 5408	43
44	0.116 8613	0.077 0091	0.050 9464	0.033 8341	44
45	0.111 2965	0.072 6501	0.047 6135	0.031 3279	45
46	0.105 9967	0.068 5378	0.044 4986	0.029 0073	46
47	0.100 9492	0.064 6583	0.041 5875	0.026 8586	47
48	0.096 1421	0.060 9984	0.038 8668	0.024 8691	48
49	0.091 5639	0.057 5457	0.036 3241	0.023 0269	49
50	0.087 2037	0.054 2884	0.033 9478	0.021 3212	50
60	0.053 5355	0.030 3143	0.017 2573	0.009 8758	60
70	0.032 8662	0.016 9274	0.008 7728	0.004 5744	70
80	0.020 1770	0.009 4522	0.004 4596	0.002 1188	80
90	0.012 3869	0.005 2780	0.002 2670	0.000 9814	90
100	0.007 6045	0.002 9472	0.001 1524	0.000 4546	100

TABLE IV—PRESENT VALUE OF 1 PER ANNUM

$$a_{\overline{n|}} = \frac{1-v^n}{i}$$

n	½%	1%	1¼%	1½%	n
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	1
2	1.985 0994	1.970 3951	1.963 1154	1.955 8834	2
3	2.970 2481	2.940 9852	2.926 5337	2.912 2004	3
4	3.950 4957	3.901 9656	3.878 0580	3.854 3846	4
5	4.925 8663	4.853 4312	4.817 8350	4.782 6450	5
6	5.896 3844	5.795 4765	5.746 0099	5.697 1872	6
7	6.862 0740	6.728 1945	6.662 7258	6.598 2140	7
8	7.822 9592	7.651 6778	7.568 1243	7.485 9251	8
9	8.779 0639	8.566 0176	8.462 3450	8.360 5173	9
10	9.730 4119	9.471 3045	9.345 5259	9.222 1846	10
11	10.677 0267	10.367 6282	10.217 8034	10.071 1178	11
12	11.618 9321	11.255 0775	11.079 3120	10.907 5052	12
13	12.556 1513	12.133 7401	11.930 1847	11.731 5322	13
14	13.488 7078	13.003 7030	12.770 5528	12.543 3815	14
15	14.416 6246	13.865 0525	13.600 5459	13.343 2330	15
16	15.339 9250	14.717 8738	14.420 2923	14.131 2640	16
17	16.258 6319	15.562 2513	15.229 9183	14.907 6493	17
18	17.172 7680	16.398 2686	16.029 5489	15.672 5609	18
19	18.082 3562	17.226 0085	16.819 3076	16.426 1684	19
20	18.987 4192	18.045 5530	17.599 3161	17.168 6388	20
21	19.887 9792	18.856 9831	18.369 6950	17.900 1367	21
22	20.784 0590	19.660 3793	19.130 5629	18.620 8244	22
23	21.675 6806	20.455 8211	19.882 0374	19.330 8614	23
24	22.562 8662	21.243 3873	20.624 2345	20.030 4054	24
25	23.445 6380	22.023 1557	21.357 2686	20.719 6112	25
26	24.324 0179	22.795 2037	22.081 2530	21.398 6317	26
27	25.198 0278	23.559 6076	22.796 2992	22.067 6175	27
28	26.067 6894	24.316 4432	23.502 5178	22.726 7167	28
29	26.933 0242	25.065 7853	24.200 0176	23.376 0756	29
30	27.794 0540	25.807 7082	24.888 9062	24.015 8380	30
31	28.650 8000	26.542 2854	25.569 2901	24.646 1458	31
32	29.503 2836	27.269 5895	26.241 2742	25.267 1387	32
33	30.351 5259	27.989 6926	26.904 9622	25.878 9544	33
34	31.195 5482	28.702 6659	27.560 4564	26.481 7285	34
35	32.035 3713	29.408 5801	28.207 8582	27.075 5946	35
36	32.871 0162	30.107 5050	28.847 2674	27.660 6843	36
37	33.702 5037	30.799 5099	29.478 7826	28.237 1274	37
38	34.529 8544	31.484 6633	30.102 5013	28.805 0516	38
39	35.353 0890	32.163 0330	30.718 5198	29.364 5829	39
40	36.172 2279	32.834 6861	31.326 9332	29.915 8452	40
41	36.987 2914	33.499 6892	31.927 8352	30.458 9608	41
42	37.798 2999	34.158 1081	32.521 3187	30.994 0500	42
43	38.605 2735	34.810 0081	33.107 4753	31.521 2316	43
44	39.408 2324	35.455 4535	33.686 3954	32.040 6222	44
45	40.207 1964	36.094 5084	34.258 1682	32.552 3372	45
46	41.002 1855	36.727 2361	34.822 8822	33.056 4898	46
47	41.793 2194	37.353 6991	35.380 6244	33.553 1920	47
48	42.580 3178	37.973 9595	35.931 4809	34.042 5536	48
49	43.363 5003	38.588 0787	36.475 5367	34.524 6834	49
50	44.142 7864	39.196 1175	37.012 8757	34.999 6881	50
60	51.725 5608	44.955 0384	42.034 5918	39.380 2689	60
70	58.939 4176	50.168 5144	46.469 6756	43.154 8718	70
80	65.802 3054	54.888 2061	50.386 6571	46.407 3235	80
90	72.331 2996	59.160 8815	53.846 0604	49.209 8545	90
100	78.542 6448	63.028 8788	56.901 3394	51.624 7037	100

TABLE IV—PRESENT VALUE OF 1 PER ANNUM—*Continued*

$$a_{\overline{n|}} = \frac{1-v^n}{i}$$

n	1 ¾%	2%	2 ½%	3%	n
1	0.982 8010	0.980 3922	0.975 6098	0.970 8738	1
2	1.948 6988	1.941 5609	1.927 4242	1.913 4697	2
3	2.897 9840	2.883 8833	2.856 0236	2.828 6114	3
4	3.830 9425	3.807 7287	3.761 9742	3.717 0984	4
5	4.747 8551	4.713 4595	4.645 8285	4.579 7072	5
6	5.648 9976	5.601 4309	5.508 1254	5.417 1914	6
7	6.534 6414	6.471 9911	6.349 3906	6.230 2830	7
8	7.405 0530	7.325 4814	7.170 1372	7.019 6922	8
9	8.260 4943	8.162 2367	7.970 8655	7.786 1089	9
10	9.101 2229	8.982 5850	8.752 0639	8.530 2028	10
11	9.927 4918	9.786 8480	9.514 2087	9.252 6241	11
12	10.739 5497	10.575 3412	10.257 7646	9.954 0040	12
13	11.537 6410	11.348 3738	10.983 1850	10.634 9553	13
14	12.322 0059	12.106 2488	11.690 9122	11.296 0731	14
15	13.092 8805	12.849 2635	12.381 3777	11.937 9351	15
16	13.850 4968	13.577 7093	13.055 0027	12.561 1020	16
17	14.595 0828	14.291 8719	13.712 1977	13.166 1185	17
18	15.326 8627	14.992 0312	14.353 3636	13.753 5131	18
19	16.046 0567	15.678 4620	14.978 8913	14.323 7991	19
20	16.752 8813	16.351 4333	15.589 1623	14.877 4749	20
21	17.447 5492	17.011 2092	16.184 5486	15.415 0241	21
22	18.130 2695	17.658 0482	16.765 4132	15.936 9166	22
23	18.801 2476	18.292 2041	17.332 1105	16.443 6084	23
24	19.460 6856	18.913 9256	17.884 9858	16.935 5421	24
25	20.108 7820	19.523 4565	18.424 3764	17.413 1477	25
26	20.745 7317	20.121 0358	18.950 6111	17.876 8424	26
27	21.371 7264	20.706 8978	19.464 0109	18.327 0315	27
28	21.986 9547	21.281 2724	19.964 8887	18.764 1082	28
29	22.591 6017	21.844 3847	20.453 5499	19.188 4546	29
30	23.185 8493	22.396 4556	20.930 2926	19.600 4414	30
31	23.769 8765	22.937 7015	21.395 4074	20.000 4285	31
32	24.343 8590	23.468 3348	21.849 1780	20.388 7655	32
33	24.907 9695	23.988 5636	22.291 8809	20.765 7918	33
34	25.462 3779	24.498 5917	22.723 7863	21.131 8367	34
35	26.007 2510	24.998 6193	23.145 1573	21.487 2201	35
36	26.542 7528	25.488 8425	23.556 2511	21.832 2525	36
37	27.069 0446	25.969 4534	23.957 3181	22.167 2354	37
38	27.586 2846	26.440 6406	24.348 6030	22.492 4616	38
39	28.094 6286	26.902 5888	24.730 3444	22.808 2151	39
40	28.594 2296	27.355 4792	25.102 7750	23.114 7720	40
41	29.085 2379	27.799 4894	25.466 1220	23.412 4000	41
42	29.567 8014	28.234 7936	25.820 6068	23.701 3592	42
43	30.042 0652	28.661 5623	26.166 4457	23.981 9021	43
44	30.508 1722	29.079 9631	26.503 8494	24.254 2739	44
45	30.966 2626	29.490 1599	26.833 0239	24.518 7125	45
46	31.416 4743	29.892 3136	27.154 1696	24.775 4491	46
47	31.858 9428	30.286 5820	27.467 4826	25.024 7078	47
48	32.293 8013	30.673 1196	27.773 1537	25.266 7066	48
49	32.721 1806	31.052 0780	28.071 3695	25.501 6569	49
50	33.141 2095	31.423 6059	28.362 3117	25.729 7640	50
60	36.963 9855	34.760 8867	30.908 6565	27.675 5637	60
70	40.177 9027	37.498 6193	32.897 8570	29.123 4214	70
80	42.879 9347	39.744 5136	34.451 8172	30.200 7634	80
90	45.151 6104	41.586 9292	35.665 7685	31.002 4071	90
100	47.061 4730	43.098 3516	36.614 1053	31.598 9053	100

TABLE IV—PRESENT VALUE OF 1 PER ANNUM—*Continued*

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	3½%	4%	4½%	4¾%	n
1	0.966 1836	0.961 5385	0.956 9378	0.954 6539	1
2	1.899 6943	1.886 0947	1.872 6678	1.866 0181	2
3	2.801 6370	2.775 0910	2.748 9644	2.736 0554	3
4	3.673 0792	3.629 8952	3.587 5257	3.566 6400	4
5	4.515 0524	4.451 8223	4.389 9767	4.359 5609	5
6	5.328 5530	5.242 1369	5.157 8725	5.116 5259	6
7	6.114 5440	6.002 0547	5.892 7009	5.839 1656	7
8	6.873 9555	6.732 7449	6.595 8861	6.529 0363	8
9	7.607 6865	7.435 3316	7.268 7905	7.187 6242	9
10	8.316 6053	8.110 8958	7.912 7182	7.816 3477	10
11	9.001 5510	8.760 4767	8.528 9169	8.416 5610	11
12	9.663 3343	9.385 0738	9.118 5808	8.989 5571	12
13	10.302 7385	9.985 6478	9.682 8524	9.536 5700	13
14	10.920 5203	10.563 1229	10.222 8253	10.058 7780	14
15	11.517 4109	11.118 3874	10.739 5457	10.557 3060	15
16	12.094 1168	11.652 2956	11.234 0150	11.033 2277	16
17	12.651 3206	12.165 6688	11.707 1914	11.487 5682	17
18	13.189 6817	12.659 2970	12.159 9918	11.921 3062	18
19	13.709 8374	13.133 9394	12.593 2936	12.335 3758	19
20	14.212 4033	13.590 3263	13.007 9364	12.730 6690	20
21	14.697 9742	14.029 1600	13.404 7239	13.108 0372	21
22	15.167 1248	14.451 1153	13.784 4248	13.468 2933	22
23	15.620 4105	14.856 8417	14.147 7749	13.812 2132	23
24	16.058 3676	15.246 9631	14.495 4784	14.140 5376	24
25	16.481 5146	15.622 0799	14.828 2090	14.453 9739	25
26	16.890 3523	15.982 7692	15.146 6114	14.753 1970	26
27	17.285 3645	16.329 5858	15.451 3028	15.038 8516	27
28	17.667 0188	16.663 0632	15.742 8735	15.311 5528	28
29	18.035 7670	16.983 7146	16.021 8885	15.571 8881	29
30	18.392 0454	17.292 0333	16.288 8885	15.820 4183	30
31	18.736 2758	17.588 4936	16.544 3910	16.057 6785	31
32	19.068 8655	17.873 5515	16.788 8909	16.284 1800	32
33	19.390 2082	18.147 6457	17.022 8621	16.500 4105	33
34	19.700 6842	18.411 1978	17.246 7580	16.706 8358	34
35	20.000 6611	18.664 6132	17.461 0124	16.903 9005	35
36	20.290 4938	18.908 2820	17.666 0406	17.092 0291	36
37	20.570 5254	19.142 5788	17.862 2398	17.271 6269	37
38	20.841 0874	19.367 8642	18.049 9902	17.443 0805	38
39	21.102 4999	19.584 4848	18.229 6557	17.606 7595	39
40	21.355 0723	19.792 7739	18.401 5844	17.763 0162	40
41	21.599 1037	19.993 0518	18.566 1095	17.912 1873	41
42	21.834 8828	20.185 6267	18.723 5498	18.054 5941	42
43	22.062 6887	20.370 7949	18.874 2103	18.190 5433	43
44	22.282 7910	20.548 8413	19.018 3830	18.320 3277	44
45	22.495 4503	20.720 0397	19.156 3474	18.444 2269	45
46	22.700 9181	20.884 6536	19.288 3707	18.562 5078	46
47	22.899 4378	21.042 9361	19.414 7088	18.675 4251	47
48	23.091 2442	21.195 1309	19.535 6065	18.783 2221	48
49	23.276 5645	21.341 4720	19.651 2981	18.886 1308	49
50	23.455 6179	21.482 1846	19.762 0078	18.984 3731	50
60	24.944 7341	22.623 4900	20.638 0220	19.752 2689	60
70	26.000 3966	23.394 5150	21.202 1119	20.235 0630	70
80	26.748 7757	23.915 3918	21.565 3449	20.538 6070	80
90	27.279 3156	24.267 2776	21.799 2408	20.729 4523	90
100	27.655 4254	24.504 9990	21.949 8527	20.849 4412	100

TABLE IV—PRESENT VALUE OF 1 PER ANNUM—*Continued*

$$a_{\overline{n|}} = \frac{1-v^n}{i}$$

n	5%	6%	7%	8%	n
1	0.952 3810	0.943 3962	0.934 5794	0.925 9259	1
2	1.859 4104	1.833 3927	1.808 0182	1.783 2648	2
3	2.723 2480	2.673 0120	2.624 3160	2.577 0970	3
4	3.545 9505	3.465 1056	3.387 2113	3.312 1268	4
5	4.329 4767	4.212 3638	4.100 1974	3.992 7100	5
6	5.075 6921	4.917 3243	4.766 5397	4.622 8797	6
7	5.786 3734	5.582 3814	5.389 2894	5.206 3701	7
8	6.463 2128	6.209 7938	5.971 2985	5.746 6389	8
9	7.107 8217	6.801 6923	6.515 2322	6.246 8879	9
10	7.721 7349	7.360 0870	7.023 5816	6.710 0814	10
11	8.306 4142	7.886 8746	7.498 6744	7.138 9643	11
12	8.863 2516	8.383 8439	7.942 6863	7.536 0780	12
13	9.393 5730	8.852 6830	8.357 6508	7.903 7759	13
14	9.898 6409	9.294 9839	8.745 4680	8.244 2370	14
15	10.379 6580	9.712 2490	9.107 9140	8.559 4787	15
16	10.837 7696	10.105 8953	9.446 6486	8.851 3692	16
17	11.274 0662	10.477 2597	9.763 2230	9.121 6381	17
18	11.689 5869	10.827 6035	10.059 0869	9.371 8871	18
19	12.085 3209	11.158 1165	10.335 5952	9.603 5992	19
20	12.462 2103	11.469 9212	10.594 0143	9.818 1474	20
21	12.821 1527	11.764 0766	10.835 5273	10.016 8032	21
22	13.163 0026	12.041 5817	11.061 2405	10.200 7437	22
23	13.488 5739	12.303 3790	11.272 1874	10.371 0590	23
24	13.798 6418	12.550 3575	11.469 3340	10.528 7583	24
25	14.093 9446	12.783 3562	11.653 5832	10.674 7762	25
26	14.375 1853	13.003 1662	11.825 7787	10.809 9780	26
27	14.643 0336	13.210 5341	11.986 7090	10.935 1648	27
28	14.898 1273	13.406 1643	12.137 1113	11.051 0785	28
29	15.141 0736	13.590 7210	12.277 6741	11.158 4060	29
30	15.372 4510	13.764 8312	12.409 0412	11.257 7833	30
31	15.592 8105	13.929 0860	12.531 8142	11.349 7994	31
32	15.802 6767	14.084 0434	12.646 5553	11.434 9994	32
33	16.002 5492	14.230 2296	12.753 7900	11.513 8884	33
34	16.192 9040	14.368 1411	12.854 0094	11.586 9337	34
35	16.374 1943	14.498 2464	12.947 6723	11.654 5682	35
36	16.546 8517	14.620 9871	13.035 2078	11.717 1928	36
37	16.711 2873	14.736 7803	13.117 0166	11.775 1785	37
38	16.867 8927	14.846 0192	13.193 4735	11.828 8690	38
39	17.017 0407	14.949 0747	13.264 9285	11.878 5824	39
40	17.159 0864	15.046 2969	13.331 7088	11.924 6133	40
41	17.294 3680	15.138 0159	13.394 1204	11.967 2346	41
42	17.423 2076	15.224 5433	13.452 4490	12.006 6987	42
43	17.545 9120	15.306 1729	13.506 9617	12.043 2395	43
44	17.662 7733	15.383 1820	13.557 9081	12.077 0736	44
45	17.774 0698	15.455 8321	13.605 5216	12.108 4015	45
46	17.880 0665	15.524 3699	13.650 0202	12.137 4088	46
47	17.981 0157	15.589 0282	13.691 6076	12.164 2674	47
48	18.077 1578	15.650 0266	13.730 4744	12.189 1365	48
49	18.168 7217	15.707 5723	13.766 7986	12.212 1634	49
50	18.255 9255	15.761 8606	13.800 7463	12.233 4846	50
60	18.929 2895	16.161 4277	14.039 1812	12.376 5518	60
70	19.342 6766	16.384 5439	14.160 3893	12.442 8196	70
80	19.596 4605	16.509 1308	14.222 0054	12.473 5144	80
90	19.752 2617	16.578 6994	14.253 3279	12.487 7320	90
100	19.847 9102	16.617 5462	14.269 2507	12.494 3176	100

TABLE V—AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	½%	1%	1¼%	1½%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.005 0000	2.010 0000	2.012 5000	2.015 0000	2
3	3.015 0250	3.030 1000	3.037 6562	3.045 2250	3
4	4.030 1001	4.060 4010	4.075 6270	4.090 9034	4
5	5.050 2506	5.101 0050	5.126 5723	5.152 2669	5
6	6.075 5019	6.152 0151	6.190 6544	6.229 5509	6
7	7.105 8794	7.213 5352	7.268 0376	7.322 9942	7
8	8.141 4088	8.285 6706	8.358 8881	8.432 8391	8
9	9.182 1158	9.368 5273	9.463 3742	9.559 3317	9
10	10.228 0264	10.462 2125	10.581 6664	10.702 7217	10
11	11.279 1665	11.566 8347	11.713 9372	11.863 2625	11
12	12.335 5624	12.682 5030	12.860 3614	13.041 2114	12
13	13.397 2402	13.809 3280	14.021 1159	14.236 8296	13
14	14.464 2264	14.947 4213	15.196 3799	15.450 3820	14
15	15.536 5475	16.096 8955	16.386 3346	16.682 1378	15
16	16.614 2303	17.257 8645	17.591 1638	17.932 3698	16
17	17.697 3014	18.430 4431	18.811 0534	19.201 3554	17
18	18.785 7879	19.614 7476	20.046 1915	20.489 3757	18
19	19.879 7168	20.810 8950	21.296 7689	21.796 7164	19
20	20.979 1154	22.019 0040	22.562 9785	23.123 6671	20
21	22.084 0110	23.239 1940	23.845 0158	24.470 5221	21
22	23.194 4311	24.471 5860	25.143 0785	25.837 5799	22
23	24.310 4032	25.716 3018	26.457 3670	27.225 1436	23
24	25.431 9552	26.973 4648	27.788 0840	28.633 5208	24
25	26.559 1150	28.243 1995	29.135 4351	30.063 0236	25
26	27.691 9106	29.525 6315	30.499 6280	31.513 9690	26
27	28.830 3702	30.820 8878	31.880 8734	32.986 6785	27
28	29.974 5220	32.129 0967	33.279 3843	34.481 4787	28
29	31.124 3946	33.450 3877	34.695 3766	35.998 7008	29
30	32.280 0166	34.784 8915	36.129 0688	37.538 6814	30
31	33.441 4167	36.132 7404	37.580 6822	39.101 7616	31
32	34.608 6238	37.494 0678	39.050 4407	40.688 2880	32
33	35.781 6669	38.869 0085	40.538 5712	42.298 6123	33
34	36.960 5752	40.257 6986	42.045 3033	43.933 0915	34
35	38.145 3781	41.660 2756	43.570 8696	45.592 0879	35
36	39.336 1050	43.076 8784	45.115 5055	47.275 9692	36
37	40.532 7855	44.507 6471	46.679 4493	48.985 1087	37
38	41.735 4494	45.952 7236	48.262 9424	50.719 8854	38
39	42.944 1267	47.412 2508	49.866 2292	52.480 6837	39
40	44.158 8473	48.886 3734	51.489 5571	54.267 8939	40
41	45.379 6415	50.375 2371	53.133 1765	56.081 9123	41
42	46.606 5397	51.878 9895	54.797 3412	57.923 1410	42
43	47.839 5724	53.397 7794	56.482 3080	59.791 9881	43
44	49.078 7703	54.931 7572	58.188 3369	61.688 8679	44
45	50.324 1642	56.481 0747	59.915 6911	63.614 2010	45
46	51.575 7850	58.045 8855	61.664 6372	65.568 4140	46
47	52.833 6639	59.626 3443	63.435 4452	67.551 9402	47
48	54.097 8322	61.222 6078	65.228 3882	69.565 2193	48
49	55.368 3214	62.834 8338	67.043 7431	71.608 6976	49
50	56.645 1630	64.463 1822	68.881 7899	73.682 8280	50
60	69.770 0305	81.669 6699	88.574 5078	96.214 6517	60
70	83.566 1055	100.676 3368	110.871 9978	122.363 7530	70
80	98.067 7136	121.671 5217	136.118 7953	152.710 8525	80
90	113.310 9358	144.863 2675	164.705 0076	187.929 9004	90
100	129.333 6984	170.481 3829	197.072 3420	228.803 0433	100

TABLE V—AMOUNT OF 1 PER ANNUM—*Continued*

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}$$

n	1¾%	2%	2½%	3%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.017 5000	2.020 0000	2.025 0000	2.030 0000	2
3	3.052 8062	3.060 4000	3.075 6250	3.090 9000	3
4	4.106 2304	4.121 6080	4.152 5156	4.183 6270	4
5	5.178 0894	5.204 0402	5.256 3285	5.309 1358	5
6	6.268 7060	6.308 1210	6.387 7367	6.468 4099	6
7	7.378 4083	7.434 2834	7.547 4302	7.662 4622	7
8	8.507 5304	8.582 9690	8.736 1159	8.892 3360	8
9	9.656 4122	9.754 6284	9.954 5188	10.159 1061	9
10	10.825 3994	10.949 7210	11.203 3818	11.463 8793	10
11	12.014 8439	12.168 7154	12.483 4663	12.807 7957	11
12	13.225 1037	13.412 0897	13.795 5530	14.192 0296	12
13	14.456 5430	14.680 3315	15.140 4408	15.617 7904	13
14	15.709 5325	15.973 9382	16.518 9528	17.086 3242	14
15	16.984 4494	17.293 4169	17.931 9267	18.598 9139	15
16	18.281 6772	18.639 2852	19.380 2248	20.156 8813	16
17	19.601 6066	20.012 0710	20.864 7304	21.761 5877	17
18	20.944 6347	21.412 3124	22.386 3487	23.414 4354	18
19	22.311 1658	22.840 5586	23.946 0074	25.116 8684	19
20	23.701 6112	24.297 3698	25.544 6576	26.870 3745	20
21	25.116 3894	25.783 3172	27.183 2740	28.676 4857	21
22	26.555 9262	27.298 9835	28.862 8559	30.536 7803	22
23	28.020 6549	28.844 9632	30.584 4273	32.452 8837	23
24	29.511 0164	30.421 8625	32.349 0380	34.426 4702	24
25	31.027 4592	32.030 2997	34.157 7639	36.459 2643	25
26	32.570 4397	33.670 9057	36.011 7080	38.553 0422	26
27	34.140 4224	35.344 3238	37.912 0007	40.709 6335	27
28	35.737 8798	37.051 2103	39.859 8008	42.930 9225	28
29	37.363 2927	38.792 2345	41.856 2958	45.218 8502	29
30	39.017 1503	40.568 0792	43.902 7032	47.575 4157	30
31	40.699 9504	42.379 4408	46.000 2707	50.002 6782	31
32	42.412 1996	44.227 0296	48.150 2775	52.502 7585	32
33	44.154 4130	46.111 5702	50.354 0344	55.077 8413	33
34	45.927 1153	48.033 8016	52.612 8853	57.730 1765	34
35	47.730 8398	49.994 4776	54.928 2074	60.462 0818	35
36	49.566 1295	51.994 3672	57.301 4126	63.275 9443	36
37	51.433 5368	54.034 2545	59.733 9479	66.174 2226	37
38	53.333 6236	56.114 9396	62.227 2966	69.159 4493	38
39	55.266 9621	58.237 2384	64.782 9791	72.234 2328	39
40	57.234 1339	60.401 9832	67.402 5535	75.401 2597	40
41	59.235 7312	62.610 0228	70.087 6174	78.663 2975	41
42	61.272 3565	64.862 2233	72.839 8078	82.023 1964	42
43	63.344 6228	67.159 4678	75.660 8030	85.483 8923	43
44	65.453 1537	69.502 6571	78.552 3231	89.048 4091	44
45	67.598 5839	71.892 7103	81.516 1312	92.719 8614	45
46	69.781 5591	74.330 5645	84.554 0344	96.501 4572	46
47	72.002 7364	76.817 1758	87.667 8853	100.396 5010	47
48	74.262 7842	79.353 5193	90.859 5824	104.408 3960	48
49	76.562 3830	81.940 5897	94.131 0720	108.540 6478	49
50	78.902 2247	84.579 4014	97.484 3488	112.796 8673	50
60	104.675 2159	114.051 5394	135.991 5900	163.053 4368	60
70	135.330 7583	149.977 9111	185.284 1142	230.594 0637	70
80	171.793 8242	193.771 9578	248.382 7126	321.363 0186	80
90	215.164 6172	247.156 6563	329.154 2533	443.348 9036	90
100	266.751 7679	312.232 3059	432.548 6540	607.287 7327	100

TABLE V—AMOUNT OF 1 PER ANNUM

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}$$

n	3½%	4%	4½%	4¾%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.035 0000	2.040 0000	2.045 0000	2.047 5000	2
3	3.106 2250	3.121 6000	3.137 0250	3.144 7562	3
4	4.214 9429	4.246 4640	4.278 1911	4.294 1322	4
5	5.362 4659	5.416 3226	5.470 7097	5.498 1034	5
6	6.550 1522	6.632 9755	6.716 8917	6.759 2634	6
7	7.779 4075	7.898 2945	8.019 1518	8.080 3284	7
8	9.051 6868	9.214 2263	9.380 0136	9.464 1440	8
9	10.368 4958	10.582 7953	10.802 1142	10.913 6908	9
10	11.731 3932	12.006 1071	12.288 2094	12.432 0911	10
11	13.141 9919	13.486 3514	13.841 1788	14.022 6154	11
12	14.601 9616	15.025 8055	15.464 0318	15.688 6897	12
13	16.113 0303	16.626 8377	17.159 9133	17.433 9024	13
14	17.676 9864	18.291 9112	18.932 1094	19.262 0128	14
15	19.295 6809	20.023 5876	20.784 0543	21.176 9584	15
16	20.971 0297	21.824 5311	22.719 3367	23.182 8640	16
17	22.705 0158	23.697 5124	24.741 7069	25.284 0500	17
18	24.499 6913	25.645 4129	26.855 0837	27.485 0424	18
19	26.357 1805	27.671 2294	29.063 5625	29.790 5819	19
20	28.279 6818	29.778 0786	31.371 4228	32.205 6345	20
21	30.269 4707	31.969 2017	33.783 1368	34.735 4022	21
22	32.328 9022	34.247 9698	36.303 3780	37.385 3338	22
23	34.460 4137	36.617 8886	38.937 0300	40.161 1371	23
24	36.666 5282	39.082 6041	41.689 1963	43.068 7911	24
25	38.949 8567	41.645 9083	44.565 2102	46.114 5587	25
26	41.313 1017	44.311 7446	47.570 6446	49.305 0002	26
27	43.759 0602	47.084 2144	50.711 3236	52.646 9877	27
28	46.290 6273	49.967 5830	53.993 3332	56.147 7197	28
29	48.910 7993	52.966 2863	57.423 0332	59.814 7363	29
30	51.622 6773	56.084 9378	61.007 0697	63.655 9363	30
31	54.429 4710	59.328 3353	64.752 3878	67.679 5933	31
32	57.334 5025	62.701 4687	68.666 2452	71.894 3740	32
33	60.341 2100	66.209 5274	72.756 2263	76.309 3567	33
34	63.453 1524	69.857 9085	77.030 2565	80.934 0512	34
35	66.674 0127	73.652 2249	81.496 6180	85.778 4186	35
36	70.007 6032	77.598 3138	86.163 9658	90.852 8935	36
37	73.457 8693	81.702 2464	91.041 3443	96.168 4059	37
38	77.028 8947	85.970 3363	96.138 2048	101.736 4052	38
39	80.724 9060	90.409 1497	101.464 4240	107.568 8845	39
40	84.550 2778	95.025 5157	107.030 3231	113.678 4065	40
41	88.509 5375	99.826 5363	112.846 6876	120.078 1308	41
42	92.607 3713	104.819 5978	118.924 7885	126.781 8420	42
43	96.848 6293	110.012 3817	125.276 4040	133.803 9795	43
44	101.238 3313	115.412 8770	131.913 8422	141.159 6685	44
45	105.781 6729	121.029 3920	138.849 9651	148.864 7528	45
46	110.484 0314	126.870 5677	146.098 2135	156.935 8285	46
47	115.350 9726	132.945 3904	153.672 6331	165.390 2804	47
48	120.388 2566	139.263 2060	161.587 9016	174.246 3187	48
49	125.601 8456	145.833 7343	169.859 3572	183.523 0188	49
50	130.997 9102	152.667 0837	178.503 0283	193.240 3622	50
60	196.516 8829	237.990 6852	289.497 9540	319.785 5885	60
70	288.937 8646	364.290 4588	461.869 6796	521.058 8495	70
80	419.306 7868	551.244 9768	729.557 6985	841.188 8678	80
90	603.205 0270	827.983 3335	1145.269 0066	1350.363 4500	90
100	862.611 6567	1237.623 7046	1790.855 9563	2160.218 0106	100

TABLE V—AMOUNT OF 1 PER ANNUM—*Continued*

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i}$$

n	5%	6%	7%	8%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.050 0000	2.060 0000	2.070 0000	2.080 0000	2
3	3.152 5000	3.183 6000	3.214 9000	3.246 4000	3
4	4.310 1250	4.374 6160	4.439 9430	4.506 1120	4
5	5.525 6312	5.637 0930	5.750 7390	5.866 6010	5
6	6.801 9128	6.975 3185	7.153 2907	7.335 9290	6
7	8.142 0084	8.393 8376	8.654 0211	8.922 8034	7
8	9.549 1089	9.897 4679	10.259 8026	10.636 6276	8
9	11.026 5643	11.491 3160	11.977 9888	12.487 5578	9
10	12.577 8925	13.180 7949	13.816 4480	14.486 5625	10
11	14.206 7872	14.971 6426	15.783 5993	16.645 4875	11
12	15.917 1265	16.869 9412	17.888 4513	18.977 1265	12
13	17.712 9828	18.882 1377	20.140 6429	21.495 2966	13
14	19.598 6320	21.015 0659	22.550 4879	24.214 9203	14
15	21.578 5636	23.275 9699	25.129 0220	27.152 1139	15
16	23.657 4918	25.672 5281	27.888 0536	30.324 2830	16
17	25.840 3664	28.212 8798	30.840 2173	33.750 2257	17
18	28.132 3847	30.905 6526	33.999 0325	37.450 2437	18
19	30.539 0039	33.759 9917	37.378 9648	41.446 2632	19
20	33.065 9541	36.785 5912	40.995 4923	45.761 9643	20
21	35.719 2518	39.992 7267	44.865 1768	50.422 9214	21
22	38.505 2144	43.392 2903	49.005 7392	55.456 7552	22
23	41.430 4751	46.995 8277	53.436 1409	60.893 2956	23
24	44.501 9989	50.815 5774	58.176 6708	66.764 7592	24
25	47.727 0988	54.864 5120	63.249 0377	73.105 9400	25
26	51.113 4538	59.156 3827	68.676 4704	79.954 4152	26
27	54.669 1264	63.705 7657	74.483 8233	87.350 7684	27
28	58.402 5828	68.528 1116	80.697 6909	95.338 8298	28
29	62.322 7119	73.639 7983	87.346 5293	103.965 9362	29
30	66.438 8475	79.058 1862	94.460 7863	113.283 2111	30
31	70.760 7899	84.801 6774	102.073 0414	123.345 8680	31
32	75.298 8294	90.889 7780	110.218 1543	134.213 5374	22
33	80.063 7708	97.343 1647	118.933 4251	145.950 6204	33
34	85.066 9594	104.183 7546	128.258 7648	158.626 6701	34
35	90.320 3074	111.434 7799	138.236 8784	172.316 8037	35
36	95.836 3227	119.120 8667	148.913 4598	187.102 1480	36
37	101.628 1389	127.268 1187	160.337 4020	203.070 3198	37
38	107.709 5458	135.904 2058	172.561 0202	220.315 9454	38
39	114.095 0231	145.058 4581	185.640 2916	238.941 2210	39
40	120.799 7742	154.761 9656	199.635 1120	259.056 5187	40
41	127.839 7630	165.047 6836	214.609 5698	280.781 0402	41
42	135.231 7511	175.950 5446	230.632 2397	304.243 5234	42
43	142.993 3387	187.507 5772	247.776 4965	329.583 0053	43
44	151.143 0056	199.758 0319	266.120 8512	356.949 6457	44
45	159.700 1559	212.743 5138	285.749 3108	386.505 6174	45
46	168.685 1637	226.508 1246	306.751 7626	418.426 0668	46
47	178.119 4218	241.098 6121	329.224 3860	452.900 1521	47
48	188.025 3929	256.564 5288	353.270 0930	490.132 1643	48
49	198.426 6626	272.958 4006	378.998 9995	530.342 7374	49
50	209.347 9957	290.335 9046	406.528 9295	573.770 1564	50
60	353.583 7179	533.128 1809	813.520 3834	1253.213 2958	60
70	588.528 5107	967.932 1696	1614.134 1742	2720.080 0738	70
80	971.228 8213	1746.599 8914	3189.062 6797	5886.935 4283	80
90	1594.607 3010	3141.075 1872	6287.185 4268	12723.938 6160	90
100	2610.025 1569	5638.368 0586	12381.661 7938	27484.515 7043	100

TABLE VI—ANNUITY WHICH 1 WILL BUY

$$\frac{1}{a_{\overline{n}|}}=\frac{1}{s_{\overline{n}|}}+i$$

n	½%	1%	1¼%	1½%	n
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1
2	0.503 7531	0.507 5124	0.509 3944	0.511 2779	2
3	0.336 6722	0.340 0221	0.341 7012	0.343 3830	3
4	0.253 1328	0.256 2811	0.257 8610	0.259 4448	4
5	0.203 0100	0.206 0398	0.207 5621	0.209 0893	5
6	0.169 5955	0.172 5484	0.174 0338	0.175 5252	6
7	0.145 7285	0.148 6283	0.150 0887	0.151 5562	7
8	0.127 8289	0.130 6903	0.132 1331	0.133 5840	8
9	0.113 9074	0.116 7404	0.118 1706	0.119 6098	9
10	0.102 7706	0.105 5821	0.107 0031	0.108 4342	10
11	0.093 6590	0.096 4541	0.097 8684	0.099 2938	11
12	0.086 0664	0.088 8488	0.090 2583	0.091 6800	12
13	0.079 6422	0.082 4148	0.083 8210	0.085 2404	13
14	0.074 1361	0.076 9012	0.078 3052	0.079 7233	14
15	0.069 3644	0.072 1238	0.073 5265	0.074 9444	15
16	0.065 1894	0.067 9446	0.069 3467	0.070 7651	16
17	0.061 5058	0.064 2581	0.065 6602	0.067 0796	17
18	0.058 2317	0.060 9820	0.062 3848	0.063 8058	18
19	0.055 3025	0.058 0518	0.059 4555	0.060 8785	19
20	0.052 6664	0.055 4153	0.056 8204	0.058 2457	20
21	0.050 2816	0.053 0308	0.054 4375	0.055 8655	21
22	0.048 1138	0.050 8637	0.052 2724	0.053 7033	22
23	0.046 1346	0.048 8858	0.050 2967	0.051 7308	23
24	0.044 3206	0.047 0735	0.048 4866	0.049 9241	24
25	0.042 6519	0.045 4068	0.046 8225	0.048 2634	25
26	0.041 1116	0.043 8689	0.045 2873	0.046 7320	26
27	0.039 6856	0.042 4455	0.043 8668	0.045 3153	27
28	0.038 3617	0.041 1244	0.042 5486	0.044 0011	28
29	0.037 1291	0.039 8950	0.041 3223	0.042 7788	29
30	0.035 9789	0.038 7481	0.040 1785	0.041 6392	30
31	0.034 9030	0.037 6757	0.039 1094	0.040 5743	31
32	0.033 8945	0.036 6709	0.038 1079	0.039 5771	32
33	0.032 9473	0.035 7274	0.037 1679	0.038 6414	33
34	0.032 0559	0.034 8400	0.036 2839	0.037 7619	34
35	0.031 2155	0.034 0037	0.035 4511	0.036 9336	35
36	0.030 4219	0.033 2143	0.034 6653	0.036 1524	36
37	0.029 6714	0.032 4680	0.033 9227	0.035 4144	37
38	0.028 9604	0.031 7615	0.033 2198	0.034 7161	38
39	0.028 2861	0.031 0916	0.032 5536	0.034 0546	39
40	0.027 6455	0.030 4556	0.031 9214	0.033 4271	40
41	0.027 0363	0.029 8510	0.031 3206	0.032 8311	41
42	0.026 4562	0.029 2756	0.030 7491	0.032 2643	42
43	0.025 9032	0.028 7274	0.030 2047	0.031 7246	43
44	0.025 3754	0.028 2044	0.029 6856	0.031 2104	44
45	0.024 8712	0.027 7050	0.029 1901	0.030 7198	45
46	0.024 3889	0.027 2278	0.028 7168	0.030 2512	46
47	0.023 9273	0.026 7711	0.028 2641	0.029 8034	47
48	0.023 4850	0.026 3338	0.027 8307	0.029 3750	48
49	0.023 0609	0.025 9147	0.027 4156	0.028 9648	49
50	0.022 6538	0.025 5127	0.027 0176	0.028 5717	50
60	0.019 3328	0.022 2444	0.023 7899	0.025 3934	60
70	0.016 9666	0.019 9328	0.021 5194	0.023 1724	70
80	0.015 1970	0.018 2188	0.019 8465	0.021 5483	80
90	0.013 8253	0.016 9031	0.018 5715	0.020 3211	90
100	0.012 7319	0.015 8657	0.017 5743	0.019 3706	100

TABLE VI—ANNUITY WHICH 1 WILL BUY—*Continued*

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	1¾%	2%	2½%	3%	n
1	1.017 5000	1.020 0000	1.025 0000	1.030 0000	1
2	0.513 1630	0.515 0495	0.518 8272	0.522 6108	2
3	0.345 0675	0.346 7547	0.350 1372	0.353 5304	3
4	0.261 0324	0.262 6238	0.265 8179	0.269 0270	4
5	0.210 6214	0.212 1584	0.215 2469	0.218 3546	5
6	0.177 0226	0.178 5258	0.181 5500	0.184 5975	6
7	0.153 0306	0.154 5120	0.157 4954	0.160 5064	7
8	0.135 0429	0.136 5098	0.139 4674	0.142 4564	8
9	0.121 0581	0.122 5154	0.125 4569	0.128 4339	9
10	0.109 8754	0.111 3265	0.114 2588	0.117 2305	10
11	0.100 7304	0.102 1779	0.105 1060	0.108 0774	11
12	0.093 1138	0.094 5596	0.097 4871	0.100 4621	12
13	0.086 6728	0.088 1184	0.091 0483	0.094 0295	13
14	0.081 1556	0.082 6020	0.085 5365	0.088 5263	14
15	0.076 3774	0.077 8255	0.080 7665	0.083 7666	15
16	0.072 1996	0.073 6501	0.076 5990	0.079 6108	16
17	0.068 5162	0.069 9698	0.072 9278	0.075 9525	17
18	0.065 2449	0.066 7021	0.069 6701	0.072 7087	18
19	0.062 3206	0.063 7818	0.066 7606	0.069 8139	19
20	0.059 6912	0.061 1567	0.064 1471	0.067 2157	20
21	0.057 3146	0.058 7848	0.061 7873	0.064 8718	21
22	0.055 1564	0.056 6314	0.059 6466	0.062 7474	22
23	0.053 1880	0.054 6681	0.057 6964	0.060 8139	23
24	0.051 3856	0.052 8711	0.055 9128	0.059 0474	24
25	0.049 7295	0.051 2204	0.054 2759	0.057 4279	25
26	0.048 2021	0.049 6992	0.052 7688	0.055 9383	26
27	0.046 7908	0.048 2931	0.051 3769	0.054 5642	27
28	0.045 4815	0.046 9897	0.050 0879	0.053 2932	28
29	0.044 2642	0.045 7784	0.048 8913	0.052 1147	29
30	0.043 1298	0.044 6499	0.047 7776	0.051 0193	30
31	0.042 0700	0.043 5964	0.046 7390	0.049 9989	31
32	0.041 0781	0.042 6106	0.045 7683	0.049 0466	32
33	0.040 1478	0.041 6865	0.044 8594	0.048 1561	33
34	0.039 2736	0.040 8187	0.044 0068	0.047 3220	34
35	0.038 4508	0.040 0022	0.043 2056	0.046 5393	35
36	0.037 6751	0.039 2328	0.042 4516	0.045 8038	36
37	0.036 9426	0.038 5068	0.041 7409	0.045 1116	37
38	0.036 2499	0.037 8206	0.041 0701	0.044 4593	38
39	0.035 5940	0.037 1711	0.040 4362	0.043 8438	39
40	0.034 9721	0.036 5558	0.039 8362	0.043 2624	40
41	0.034 3817	0.035 9719	0.039 2679	0.042 7124	41
42	0.033 8206	0.035 4173	0.038 7288	0.042 1917	42
43	0.033 2867	0.034 8899	0.038 2169	0.041 6981	43
44	0.032 7781	0.034 3879	0.037 7304	0.041 2298	44
45	0.032 2932	0.033 9096	0.037 2675	0.040 7852	45
46	0.031 8304	0.033 4534	0.036 8268	0.040 3625	46
47	0.031 3884	0.033 0179	0.036 4067	0.039 9605	47
48	0.030 9657	0.032 6018	0.036 0060	0.039 5778	48
49	0.030 5612	0.032 2040	0.035 6235	0.039 2131	49
50	0.030 1739	0.031 8232	0.035 2581	0.038 8655	50
60	0.027 0534	0.028 7680	0.032 3534	0.036 1330	60
70	0.024 8893	0.026 6676	0.030 3971	0.034 3366	70
80	0.023 3209	0.025 1607	0.029 0260	0.033 1118	80
90	0.022 1476	0.024 0460	0.028 0381	0.032 2556	90
100	0.021 2488	0.023 2027	0.027 3119	0.031 6467	100

TABLE VI—ANNUITY WHICH 1 WILL BUY—*Continued*

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i$$

n	3½%	4%	4½%	4¾%	n
1	1.035 0000	1.040 0000	1.045 0000	1.047 5000	1
2	0.526 4005	0.530 1961	0.533 9976	0.535 9005	2
3	0.356 9342	0.360 3485	0.363 7734	0.365 4897	3
4	0.272 2511	0.275 4900	0.278 7436	0.280 3759	4
5	0.221 4814	0.224 6271	0.227 7916	0.229 3809	5
6	0.187 6682	0.190 7619	0.193 8784	0.195 4451	6
7	0.163 5445	0.166 6096	0.169 7015	0.171 2574	7
8	0.145 4766	0.148 5278	0.151 6096	0.153 1620	8
9	0.131 4460	0.134 4930	0.137 5745	0.139 1280	9
10	0.120 2414	0.123 2909	0.126 3788	0.127 9370	10
11	0.111 0920	0.114 1490	0.117 2482	0.118 8134	11
12	0.103 4840	0.106 5522	0.109 6662	0.111 2402	12
13	0.097 0616	0.100 1437	0.103 2754	0.104 8595	13
14	0.091 5707	0.094 6690	0.097 8203	0.099 4156	14
15	0.086 8251	0.089 9411	0.093 1138	0.094 7211	15
16	0.082 6848	0.085 8200	0.089 0154	0.090 6353	16
17	0.079 0431	0.082 1985	0.085 4176	0.087 0506	17
18	0.075 8168	0.078 9933	0.082 2369	0.083 8834	18
19	0.072 9403	0.076 1386	0.079 4073	0.081 0677	19
20	0.070 3611	0.073 5818	0.076 8761	0.078 5505	20
21	0.068 0366	0.071 2801	0.074 6006	0.076 2891	21
22	0.065 9321	0.069 1988	0.072 5456	0.074 2485	22
23	0.064 0188	0.067 3091	0.070 6825	0.072 3997	23
24	0.062 2728	0.065 5868	0.068 9870	0.070 7187	24
25	0.060 6740	0.064 0120	0.067 4390	0.069 1851	25
26	0.059 2054	0.062 5674	0.066 0214	0.067 7819	26
27	0.057 8524	0.061 2385	0.064 7195	0.066 4944	27
28	0.056 6026	0.060 0130	0.063 5208	0.065 3102	28
29	0.055 4454	0.058 8799	0.062 4146	0.064 2183	29
30	0.054 3713	0.057 8301	0.061 3915	0.063 2095	30
31	0.053 3724	0.056 8554	0.060 4434	0.062 2755	31
32	0.052 4415	0.055 9486	0.059 5632	0.061 4093	32
33	0.051 5724	0.055 1036	0.058 7445	0.060 6046	33
34	0.050 7597	0.054 3148	0.057 9819	0.059 8557	34
35	0.049 9984	0.053 5773	0.057 2704	0.059 1579	35
36	0.049 2842	0.052 8869	0.056 6058	0.058 5068	36
37	0.048 6132	0.052 2396	0.055 9840	0.057 8984	37
38	0.047 9821	0.051 6319	0.055 4017	0.057 3293	38
39	0.047 3878	0.051 0608	0.054 8557	0.056 7964	39
40	0.046 8273	0.050 5235	0.054 3432	0.056 2968	40
41	0.046 2982	0.050 0174	0.053 8616	0.055 8279	41
42	0.045 7983	0.049 5402	0.053 4087	0.055 3876	42
43	0.045 3254	0.049 0899	0.052 9824	0.054 9736	43
44	0.044 8777	0.048 6645	0.052 5807	0.054 5842	44
45	0.044 4534	0.048 2625	0.052 2020	0.054 2175	45
46	0.044 0511	0.047 8820	0.051 8447	0.053 8720	46
47	0.043 6692	0.047 5219	0.051 5073	0.053 5463	47
48	0.043 3065	0.047 1806	0.051 1886	0.053 2390	48
49	0.042 9617	0.046 8571	0.050 8872	0.052 9489	49
50	0.042 6337	0.046 5502	0.050 6022	0.052 6749	50
60	0.040 0886	0.044 2018	0.048 4543	0.050 6271	60
70	0.038 4610	0.042 7451	0.047 1651	0.049 4192	70
80	0.037 3849	0.041 8141	0.046 3707	0.048 6888	80
90	0.036 6578	0.041 2078	0.045 8732	0.048 2405	90
100	0.036 1593	0.040 8080	0.045 5584	0.047 9629	100

TABLE VI—ANNUITY WHICH 1 WILL BUY—*Continued*

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i$$

n	5%	6%	7%	8%	n
1	1.050 0000	1.060 0000	1.070 0000	1.080 0000	1
2	0.537 8049	0.545 4369	0.553 0918	0.560 7692	2
3	0.367 2086	0.374 1098	0.381 0517	0.388 0335	3
4	0.282 0118	0.288 5915	0.295 2281	0.301 9208	4
5	0.230 9748	0.237 3964	0.243 8907	0.250 4564	5
6	0.197 0175	0.203 3626	0.209 7958	0.216 3154	6
7	0.172 8198	0.179 1350	0.185 5532	0.192 0724	7
8	0.154 7218	0.161 0359	0.167 4678	0.174 0148	8
9	0.140 6901	0.147 0222	0.153 4865	0.160 0797	9
10	0.129 5046	0.135 8680	0.142 3775	0.149 0295	10
11	0.120 3889	0.126 7929	0.133 3569	0.140 0763	11
12	0.112 8254	0.119 2770	0.125 9020	0.132 6950	12
13	0.106 4558	0.112 9601	0.119 6508	0.126 5218	13
14	0.101 0240	0.107 5849	0.114 3449	0.121 2968	14
15	0.096 3423	0.102 9628	0.109 7946	0.116 8295	15
16	0.092 2699	0.098 9521	0.105 8576	0.112 9769	16
17	0.088 6991	0.095 4448	0.102 4252	0.109 6294	17
18	0.085 5462	0.092 3565	0.099 4126	0.106 7021	18
19	0.082 7450	0.089 6209	0.096 7530	0.104 1276	19
20	0.080 2426	0.087 1846	0.094 3929	0.101 8522	20
21	0.077 9961	0.085 0046	0.092 2890	0.099 8322	21
22	0.075 9705	0.083 0456	0.090 4058	0.098 0321	22
23	0.074 1368	0.081 2785	0.088 7139	0.096 4222	23
24	0.072 4709	0.079 6790	0.087 1890	0.094 9780	24
25	0.070 9525	0.078 2267	0.085 8105	0.093 6788	25
26	0.069 5643	0.076 9044	0.084 5610	0.092 5071	26
27	0.068 2919	0.075 6972	0.083 4257	0.091 4481	27
28	0.067 1225	0.074 5926	0.082 3919	0.090 4889	28
29	0.066 0455	0.073 5796	0.081 4486	0.089 6185	29
30	0.065 0514	0.072 6489	0.080 5864	0.088 8274	30
31	0.064 1321	0.071 7922	0.079 7969	0.088 1073	31
32	0.063 2804	0.071 0023	0.079 0729	0.087 4508	32
33	0.062 4900	0.070 2729	0.078 4081	0.086 8516	33
34	0.061 7554	0.069 5984	0.077 7967	0.086 3041	34
35	0.061 0717	0.068 9739	0.077 2340	0.085 8033	35
36	0.060 4345	0.068 3948	0.076 7153	0.085 3447	36
37	0.059 8398	0.067 8574	0.076 2368	0.084 9244	37
38	0.059 2842	0.067 3581	0.075 7950	0.084 5389	38
39	0.058 7646	0.066 8938	0.075 3868	0.084 1851	39
40	0.058 2782	0.066 4615	0.075 0091	0.083 8602	40
41	0.057 8223	0.066 0589	0.074 6596	0.083 5615	41
42	0.057 3947	0.065 6834	0.074 3359	0.083 2868	42
43	0.056 9933	0.065 3331	0.074 0359	0.083 0341	43
44	0.056 6162	0.065 0061	0.073 7577	0.082 8015	44
45	0.056 2617	0.064 7005	0.073 4996	0.082 5873	45
46	0.055 9282	0.064 4148	0.073 2600	0.082 3899	46
47	0.055 6142	0.064 1477	0.073 0374	0.082 2080	47
48	0.055 3184	0.063 8977	0.072 8307	0.082 0403	48
49	0.055 0396	0.063 6636	0.072 6385	0.081 8856	49
50	0.054 7767	0.063 4443	0.072 4598	0.081 7429	50
60	0.052 8282	0.061 8757	0.071 2292	0.080 7980	60
70	0.051 6992	0.061 0331	0.070 6195	0.080 3676	70
80	0.051 0296	0.060 5725	0.070 3136	0.080 1699	80
90	0.050 6271	0.060 3184	0.070 1590	0.080 0786	90
100	0.050 3831	0.060 1774	0.070 0808	0.080 0364	100

TABLE VII—AMOUNT OF 1 FOR PARTS OF A YEAR

$$s=(1+i)^{\frac{1}{p}}$$

p	½%	1%	1¼%	1½%	p
2	1.002 4969	1.004 9876	1.006 2306	1.007 4721	2
4	1.001 2477	1.002 4907	1.003 1105	1.003 7291	4
12	1.000 4157	1.000 8295	1.001 0357	1.001 2415	12
p	1¾%	2%	2½%	3%	p
2	1.008 7121	1.009 9505	1.012 4228	1.014 8892	2
4	1.004 3466	1.004 9629	1.006 1922	1.007 4171	4
12	1.001 4468	1.001 6516	1.002 0598	1.002 4663	12
p	3½%	4%	4½%	4¾%	p
2	1.017 3495	1.019 8039	1.022 2524	1.023 4745	2
4	1.008 6374	1.009 8534	1.011 0650	1.011 6692	4
12	1.002 8709	1.003 2737	1.003 6748	1.003 8747	12
p	5%	6%	7%	8%	p
2	1.024 6951	1.029 5630	1.034 4080	1.039 2305	2
4	1.012 2722	1.014 6738	1.017 0585	1.019 4265	4
12	1.004 0741	1.004 8676	1.005 6541	1.006 4340	12

TABLE VIII—VALUES OF $j_{(p)}=p[(1+i)^{\frac{1}{p}}-1]$

p	½%	1%	1¼%	1½%	p
2	0.004 9938	0.009 9751	0.012 4612	0.014 9442	2
4	0.004 9907	0.009 9627	0.012 4418	0.014 9164	4
12	0.004 9886	0.009 9545	0.012 4290	0.014 8978	12
p	1¾%	2%	2½%	3%	p
2	0.017 4241	0.019 9010	0.024 8457	0.029 7783	2
4	0.017 3863	0.019 8517	0.024 7690	0.029 6683	4
12	0.017 3612	0.019 8190	0.024 7180	0.029 5952	12
p	3½%	4%	4½%	4¾%	p
2	0.034 6990	0.039 6078	0.044 5048	0.046 9489	2
4	0.034 5498	0.039 4136	0.044 2600	0.046 6766	4
12	0.034 4508	0.039 2849	0.044 0977	0.046 4962	12
p	5%	6%	7%	8%	p
2	0.049 3902	0.059 1260	0.068 8161	0.078 4610	2
4	0.049 0889	0.058 6954	0.068 2341	0.077 7062	4
12	0.048 8895	0.058 4106	0.067 8497	0.077 2084	12

TABLE IX—VALUES OF $\frac{i}{j_{(p)}}$

p	½%	1%	1¼%	1½%	p
2	1.001 2415	1.002 4938	1.003 1153	1.003 7360	2
4	1.001 8635	1.003 7422	1.004 6754	1.005 6076	4
12	1.002 2852	1.004 5751	1.005 7163	1.006 8565	12
p	1¾%	2%	2½%	3%	p
2	1.004 3560	1.004 9753	1.006 2114	1.007 4446	2
4	1.006 5388	1.007 4691	1.009 3268	1.011 1807	4
12	1.007 9957	1.009 1339	1.011 4073	1.013 6766	12
p	3½%	4%	4½%	4¾%	p
2	1.008 6748	1.009 9020	1.011 1262	1.011 7383	2
4	1.013 0309	1.014 8774	1.016 7203	1.017 6405	4
12	1.015 9420	1.018 2035	1.020 4611	1.021 5889	12
p	5%	6%	7%	8%	p
2	1.012 3475	1.014 7815	1.017 2040	1.019 6148	2
4	1.018 5594	1.022 2269	1.025 8800	1.029 5189	4
12	1.022 7148	1.027 2107	1.031 6914	1.036 1567	12

TABLE X—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x	Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x
10	100,000	749	0.007 490	0.992 510	53	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1,199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60,779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1,546	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	61	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	64	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1,980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2,158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2,391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2,487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2,505	0.080 178	0.919 822
31	84,721	721	0.008 510	0.991 490	74	28,738	2,501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2,476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2,291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2,196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 766	80	14,474	2,091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1,964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 466	0.467 534
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1,001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE XI—COMMUTATION COLUMNS

American Experience, 3½%

Age x	D_x	N_x	M_x	Age x	D_x	N_x	M_x
10	70 891.9	1 575 535.3	17 612.91	**53**	10 787.4	145 915.7	5 853.095
11	67 981.5	1 504 643.4	17 099.89	**54**	10 252.4	135 128.2	5 682.861
12	65 189.0	1 436 661.9	16 606.20	**55**	9 733.40	124 875.8	5 510.544
13	62 509.4	1 371 472.9	16 131.12	**56**	9 229.60	115 142.4	5 335.898
14	59 938.4	1 308 963.5	15 673.96	**57**	8 740.17	105 912.8	5 158.573
15	54 471.6	1 249 025.0	15 234.05	**58**	8 264.44	97 172.64	4 978.405
16	55 104.2	1 191 553.4	14 810.17	**59**	7 801.83	88 908.20	4 795.266
17	52 832.9	1 136 449.2	14 402.30	**60**	7 351.65	81 106.38	4 608.926
18	50 653.9	1 083 616.2	14 009.83	**61**	6 913.44	73 754.73	4 419.322
19	48 562.8	1 032 962.4	13 631.68	**62**	6 486.75	66 841.28	4 226.413
20	46 556.2	984 399.6	13 267.32	**63**	6 071.27	60 354.54	4 030.296
21	44 630.8	937 843.3	12 916.25	**64**	5 666.85	54 283.27	3 831.187
22	42 782.8	893 212.5	12 577.53	**65**	5 273.33	48 616.41	3 629.300
23	41 009.2	850 429.7	12 250.71	**66**	4 890.55	43 343.08	3 424.843
24	39 307.1	809 420.5	11 935.38	**67**	4 518.65	38 452.53	3 218.321
25	37 673.6	770 113.4	11 631.14	**68**	4 157.82	33 933.88	3 010.299
26	36 106.1	732 439.8	11 337.59	**69**	3 808.32	29 776.06	2 801.396
27	34 601.5	696 333.7	11 053.97	**70**	3 470.67	25 967.74	2 592.538
28	33 157.4	661 733.2	10 779.94	**71**	3 145.43	22 497.07	2 384.657
29	31 771.3	628 574.8	10 515.18	**72**	2 833.42	19 351.64	2 179.018
30	30 440.8	596 803.6	10 259.02	**73**	2 535.75	16 518.22	1 977.167
31	29 163.5	566 362.9	10 011.17	**74**	2 253.57	13 982.47	1 780.731
32	27 937.5	537 199.3	9 771.375	**75**	1 987.87	11 728.90	1 591.240
33	26 760.5	509 261.8	9 539.044	**76**	1 739.39	9 741.028	1 409.988
34	25 630.1	482 501.3	9 313.638	**77**	1 508.63	8 001.633	1 238.047
35	24 544.7	456 871.2	9 094.955	**78**	1 295.73	6 492.999	1 076.158
36	23 502.5	432 326.5	8 882.798	**79**	1 100.65	5 197.271	924.893 7
37	22 501.4	408 824.0	8 676.415	**80**	923.338	4 096.624	784.804 6
38	21 539.7	386 322.6	8 475.658	**81**	763.234	3 173.286	655.924 5
39	20 615.5	364 782.9	8 279.860	**82**	620.465	2 410.052	538.965 7
40	19 727.4	344 167.4	8 088.915	**83**	494.995	1 789.587	434.477 6
41	18 873.6	324 440.0	7 902.231	**84**	386.641	1 294.592	342.862 4
42	18 052.9	305 566.3	7 719.738	**85**	294.610	907.951 3	263.905 9
43	17 263.6	287 513.4	7 540.910	**86**	217.598	613.341 7	196.856 9
44	16 504.4	270 249.8	7 365.489	**87**	154.383	395.743 8	141.000 3
45	15 773.6	253 745.5	7 192.809	**88**	103.963	241.360 9	95.801 07
46	15 070.0	237 971.9	7 022.682	**89**	65.623 1	137.397 8	60.976 81
47	14 392.1	222 901.9	6 854.337	**90**	38.304 7	71.774 70	35.877 55
48	13 738.5	208 509.8	6 687.466	**91**	20.186 9	33.470 01	19.055 09
49	13 107.9	194 771.3	6 521.419	**92**	9.118 89	13.283 09	8.669 695
50	12 498.6	181 663.4	6 355.436	**93**	3.222 36	4.164 21	3.081 545
51	11 909.6	169 164.7	6 189.012	**94**	0.827 611	0.941 84	0.795 762
52	11 339.5	157 255.2	6 021.696	**95**	0.114 232	0.114 23	0.110 369

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